GENERALIZED SCALE-SELECTION

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ABSTRACT

Structure in digitized images resides within two scales, the inner and outer scale. The inner scale is defined by the sampling resolution, and the outer scale is given by the image size. However, some images contain almost no fine scale structure, and these may be down-sampled without essential loss of image detail. Likewise some images may be reduced in size by removing borders with no structure. Hence we define essential inner and outer scales. Such considerations are the essence of local size estimation: A textured patch in an image has an essential inner scale related to the structure of the primitive textons, and an essential outer scale given by the size of the patch. In this paper, several functionals are examined that automatically find both the essential inner and outer scales in local neighborhoods of an image.

In this preliminary work we present a general formulation for local scale selection, that is shown to be a generalization of Lindeberg’s Blob-detector and its morphological equivalent, and we present promising results using locally orderless images.

Keywords: Linear Scale-Space, Gaussian Windows, Lyaponov Functionals, Blob-Detection, Pseudo-Linear Scale-Spaces, Locally Orderless Images, Soft Histograms.

1. IMAGE STRUCTURE

At least three views of image structure exist in the image processing community: mathematical morphology, in which the shapes of the image isophotes are studied, diffusion scale-spaces that examine the linear or non-linear mean value of neighboring pixels, and statistical methods that study the full distribution of local neighborhoods. Seemingly far apart, these views are currently converging. Florack et al. [1] have shown that morphological erosion and dilation scale-spaces and the linear-diffusion scale-space are related through a non-linear transformation of the gray-values. Additionally, Koenderink et al. [2] have introduced the notion of locally disorderly images that facilitates the availability of the simple Gaussian structure for local statistical analysis. The goal of this paper is to reveal the relation between Florack’s gray-value transforms and Koenderink’s local soft histograms, and to suggest a generalized scale-selection mechanism.

2. SCALE-SPACES SIMPLIFY STRUCTURE

Scale-spaces seem to be a key tool for studying the structure of images. A scale-space is a stack of increasingly simpler and simpler images. A central scale-space of this article is the Linear scale-space, which is given by the heat diffusion equation,

$$\frac{\partial}{\partial t} L(x; t) = \Delta L(x; t) = \sum_i \frac{\partial^2}{\partial x_i^2} L(x; t),$$

with \(I(x)\) being the original image, \(x = \{x_i\} \in \Omega\) is the vector of space coordinates in the spatial domain, and \(t\) is the scale or time parameter. The Green’s function of (1) is the Gaussian kernel, thus \(L(x; t) = (4\pi t)^{-1/2} \exp \left(-\|x\|^2/(4t)\right) * I(x)\) with “*” the convolution operator and \(\| \cdot \|_2\) the Euclidean norm.

The notion of simpler structure can be made concrete through Lyaponov Functionals [3],

$$S(t) = \sum_{x \in \Omega} \Phi \left( L(x; t) \right),$$

with \(\Phi\) being a convex function, i.e. \(\Phi'' > 0\). The Lyaponov functionals are monotonic in the linear and in many non-linear scale-spaces in compliance with the gradual reduction of structure and they seem to be intimately related to the size of dominating image structure [4].

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3. LOCAL LYAPONOV FUNCTIONALS

Lyaponov functionals can be used to perform global analysis of images but for local analysis of structure, processing must be restricted to a limited area. As recently advocated [2], Gaussian (soft) windows have the simplest structure, since they uniquely guarantee not to introduce spurious detail for increasing window size. For the above mentioned reasons, local Lyaponov functionals are defined as follows:

\[ S(x; t, w) = \sum_{y \in \Omega} G(x - y; w) \Phi \left( L(y; t) \right) \]
\[ = G(x; w) \ast \Phi \left( L(x; t) \right). \]  

(3)

Although the convexity requirement on \( \Phi \) could be retained, it does not seem to be of particular advantage. Thus in the following, the basic properties of the local Lyaponov functionals will be discussed for the cases of \( \Phi \) being convex and \( \Phi \) being monotonic.

For an image of \( k \) pixels and convex \( \Phi \)'s, the limits of the local functionals w.r.t. \( w \) are essentially the global Lyaponov functionals and \( \Phi(L) \):

\[ \lim_{w \to 0} S(x; t, w) = \frac{1}{k} S(t), \]  
\[ \lim_{w \to \infty} S(x; t, w) = \Phi \left( L(x; t) \right). \]  

(4) (5)

In contrast to global Lyaponov functionals, local Lyaponov functionals are non-monotonic for fixed \( w \) and increasing \( t \), since structure moves independently of \( w \).

Regardless of \( \Phi \), the structure of \( S(x; t, w) \) by \( w \) is given by the diffusion equation:

\[ \frac{\partial}{\partial w} S(x; t, w) = \Delta S(x; t, w), \]

and since the mean value is invariant under diffusion, so is the sum. Hence the sum of local Lyaponov functionals is equal to global Lyaponov functionals:

\[ \sum_{x \in \Omega} S(x, t, w) = S(t). \]

When \( \Phi \) is monotonic, the structure of \( S(x; t, w) \) w.r.t. \( w \) is essentially given by the pseudo-linear diffusion equation [1]. That is, the ‘re-normalized’ functional \( L^*(x; t, w) = \Phi^{-1}(S(x; t, w)) \) evolves as:

\[ \frac{\partial}{\partial w} L^*(x; t, w) = \Delta L^*(x; t, w) + \mu(L) \| \nabla L^*(x; t, w) \|^2, \]

(6)

where ‘\( \nabla \)’ is the gradient operator, and \( \mu(v) = \frac{\partial^2 \Phi(v)}{\partial v^2} / \Phi(v) \) is the non-linearity parameter.

4. LYAPONOV FUNCTIONALS AND THE HISTOGRAM

In this section it will be shown that a specific class of local Lyaponov functionals is a transform of local (soft) histograms [2]. The global relation is reviewed, and this is used as a skeleton for the following proof of the local relation.

Let us consider as an example, \( \Phi(v) = v^k \) for \( k = 0 \ldots N \), where \( N \) is the number of gray-values of \( L \) (typically 255). Since space and values may be interchanged in the summation of (2), the equation may be rewritten as,

\[ S(t) = \sum_{i=0}^{N-1} \Phi(v_i) f_i, \]

where \( f_i \) is the number of points in \( L \) having the value \( v_i \). The function \{\( f_i \)\} is typically denoted the gray-value histogram. With this formulation, all the global Lyaponov functions \( S_\kappa \) may be written as,

\[ \begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ v_1 & v_2 & \ldots & v_N \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{N-1} & v_2^{N-1} & \ldots & v_N^{N-1} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix}, \]

(7)

where the matrix \{\( v_i^{N-1} \)\} is a Vandermonde matrix independent on \( t \). Since Vandermonde matrices are invertible in principle, it may be concluded that there is a one-to-one relation between the class of global Lyaponov functionals, \{\( S_\kappa \)\}, and the gray-value histogram.

The same one-to-one mapping exists for the class of local Lyaponov functionals and local (soft) histograms [2]. A local histogram \( H \) of an image \( L(x; t) \) in Linear scale-space (1) is defined as

\[ H(v; x, t, u, w) = \sum_{y \in \Omega} G(x - y; w) M(v, y; t, u), \]

(8)

where \( v \) denotes the isophote value. The function \( 0 \leq M \leq 1 \) is the soft indicator function [5],

\[ M(v, x; t, u) = \exp \left( -\frac{(L(x; t) - v)^2}{4u} \right), \]

in which each isophote \( i \) has width proportional to \( u \). In the special case treated here, \( u \to 0 \) and \( M \in \{0, 1\} \) become the regular indicator function.

A general method to show that the local histogram is a transform of at least one family of local Lyaponov functionals, is to substitute \( H(i; x, t, 0, w) \) into (7) in place of \( f_i \).

Firstly, when substituting \( H(i; x, t, 0, w) \) into (7) in place of \( f_i \) and for fixed \( x, t, w \) we get local Lyaponov func-
tionals regardless of $\Phi$, since

$$S^*(x; t, w) = \sum_{i=0}^{N-1} \Phi(v_i)H(v_i; x, t, 0, w) = \sum_{i=0}^{N-1} \Phi(v_i) \sum_{y \in \Omega} G(x - y; w)M(v_i, y; t, 0) = \sum_{i=0}^{N-1} \sum_{y \in \Omega} G(x - y; w)\Phi(v_i)M(v_i, y; t, 0).$$

The above equation may be rewritten as a sum over space instead of value, resulting in

$$S^*(x; t, w) = \sum_{y \in \Omega} G(x - y; w)\Phi(L(y; t)) = S(x; t, w).$$

This proves that local Lyaponov functionals are obtained directly from local soft histograms.

Finally, it is observed that the system matrix in (7) is independent on the soft histogram regardless of $\Phi$. Hence, when the system matrix is invertible, then there is a one-to-one relation between a family of local Lyaponov functionals and local soft Histograms. As previously discussed, that is the case for $\Phi_n(v) = v^n$. This concludes the proof.

### 5. SCALE-SELECTORS

In this section, a generalization of scale-selection is introduced, that is directly related to Lindeberg’s Blob-detector [6] and the global scale-selection of Sporring and Weickert [4]. It will further be used to introduce the morphological equivalent of blob-detection and a detector of size of statistical stable regions.

In view of local Lyaponov functionals, local scale-selection is the process of choosing the pair $t$ and $w$ for a selected point curve $x$. The simplest choice is scale-selection by extrema of the gradient magnitude in the normal scale-parameters $\log t$ and $\log w$, henceforth called the scale-selector:

$$c(x; t, w) = \sqrt{\left(\frac{\partial S(x; t, w)}{\partial \log t}\right)^2 + \left(\frac{\partial S(x; t, w)}{\partial \log w}\right)^2}. \tag{10}$$

This formulation is a generalization of well-known scale-selection methods and will further be used to suggest a number of new scale-selectors.

#### 5.1. Relations to Lindeberg’s Blob-Detector

When $\Phi(v) = av + b$ for some constants $a$ and $b$, the non-linearity parameter (6) is zero, i.e $\mu = 0$, and the pseudo-linear scale-space degenerates to a linear scale-space $S(x; t, w) = L(x; t + w)$. In this case the function $c$ is simplified to $c(x; t, w) = c(x; t') = \sqrt{2} |\frac{\partial S(x; t')}{\partial \log t}|$, where $t' = t + w$. Maximizing $c$ over $x$ and $t'$ yields Lindeberg’s blob-detector [6].

Two scenarios of scale-selection with (10) have simple relations with the Blob-Detector, and serve as generalization of the two examples given in [6, pp.326–327].

Firstly, consider the one dimensional example of a cosine embedded in the Linear scale-space:

$$f(x, t) = a \exp(-\nu^2 t) \cos(\nu x)$$

with $\nu > 0$. In the case of $w \to 0$ and $\Phi(v) = v^n$, the local Lyaponov functional is given as,

$$S_n(x; t, 0) = f(x, t)^n.$$

The function $f$ has a maximum at $\nu x = 0$ for all values of $t$, and in this point the scale-selector is given as,

$$c_n = -\sqrt{2}n\nu^2 t e^{-n\nu^2 t}.$$

Differentiation with respect to $t$ reveals a maximum at $t = 1/(n\nu^2)$. Hence the order $n$ of $\Phi$ only scales the selection.

Secondly, consider the exponential function,

$$f(x) = \exp(-|x|^2).$$

In the case of $t \to 0$ and $\Phi(v) = v^n$, the local Lyaponov functional is given as,

$$S_n(x; 0, w) = G(x; w) * f(x, t_0)^n = G(x; w) * \exp(-n|x|^2).$$

If $n = 1$, the semigroup property of the Gaussian ensures that the scale-selector will choose $w = 0$ in the spatial maximum at $x = 0$. It is clear that $n$ works as a scaling of the $x$-axis, hence $f(x, t_0)^n = f(nx; t_0)$, or conversely,

$$S_n(x; 0, w) = G(x; nw) * f(x, t_0).$$

The scale-selector will therefore choose $w = 0$ independently on $n$.

#### 5.2. Relations to other Blob-Detectors

In this section a simple relation to a global scale-selection presented earlier will be presented, and further two new methods based on morphology and statistical stability will be introduced.

The family of Lyaponov functionals defined by the Rényi Entropies was studied in [4],

$$R_\alpha(t) = \frac{1}{1 - \alpha} \log \sum_{x \in \Omega} p(x; t)^\alpha,$$

$$p(x; t) = \frac{L(x; t)}{\sum_{y \in \Omega} L(y; t)}. \tag{11}$$
For convex $\Phi$, the local Lyaponov Functionals trivially go to the global Lyaponov Functionals as $w \to \infty$ (5). Hence, for fixed $w = \infty$, maxima of $c$ are identical to the scale-selection method presented in [4].

The morphological scale-spaces of dilation and erosion are obtained in the pseudo-linear scale-spaces (6), when $\mu \to \pm \infty$. Hence, for fixed $t$ and appropriate $\Phi$ for which $\mu = \infty$, the maxima of $c$ will be the morphological equivalent of blob-detection.

Finally, a statistical blob-detector can be defined by examining the local histogram (8). For sufficient regular textured patches, the local histogram will remain unchanged as long as the window function is smaller than the size of the patch. Hence, a zero order statistical blob can be found by examining the change of the histogram as a function of log $w$ as illustrated in Figure 1. In these experiments the entropy of the local histogram was traced in a fixed position and $t = 312$ for the left and $t = 4.5$ for the right. Both used $u = 32$ and $w$ was sampled logarithmically in 16 steps. As can be observed, the area of apples is found in left and the cherry plus the area of cherries is found in the right.

6. SUMMARY

In this paper we have introduced the notion of local Lyaponov functionals. These functionals form an extension of global Lyaponov functionals, and it is shown that they are related to the local soft histograms in the most natural way. Based on the local Lyaponov functionals, a generalized scale-selector is defined and shown to be directly related to Lindeberg’s blob-detector, Sporring and Weickert’s global scale-selectors, the morphological equivalent of the blob-detector, and a detector of statistical stability.

7. REFERENCES


