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# RELATIVE COMMUTANTS OF STRONGLY SELF-ABSORBING C\*-ALGEBRAS

ILIJAS FARAH, BRADD HART, MIKAEL RØRDAM, AND AARON TIKUISIS

ABSTRACT. The relative commutant  $A' \cap A^{\mathcal{U}}$  of a strongly self-absorbing algebra  $A$  is indistinguishable from its ultrapower  $A^{\mathcal{U}}$ . This applies both to the case when  $A$  is the hyperfinite  $\text{II}_1$  factor and to the case when it is a strongly self-absorbing C\*-algebra. In the latter case we prove analogous results for  $\ell_\infty(A)/c_0(A)$  and reduced powers corresponding to other filters on  $\mathbb{N}$ . Examples of algebras with approximately inner flip and approximately inner half-flip are provided, showing the optimality of our results. We also prove that strongly self-absorbing algebras are smoothly classifiable, unlike the algebras with approximately inner half-flip.

Most uses of ultrapowers in operator algebras and elsewhere rely on two of their model-theoretic properties: countable saturation and Loś' theorem, stating that the canonical embedding of a structure in its ultrapower is elementary.<sup>1</sup> These conditions – saturation and elementary equivalence – characterize the ultrapower under the assumption of the continuum hypothesis: two elementary extensions of density character  $\aleph_1$  of a separable structure  $A$  which are saturated are necessarily isomorphic over  $A$ . This is the uniqueness theorem for saturated models; see §1.1.2 and in particular Theorem 1.2. For an arbitrary metric structure  $A$ , we can identify  $A$  with its diagonal image in an ultrapower  $A^{\mathcal{U}}$  and provided that  $A$  has a multiplicative structure consider the *central sequence algebra* (also known as the *relative commutant* of  $A$  in its ultrapower)

$$A' \cap A^{\mathcal{U}} := \{b \in A^{\mathcal{U}} : ab = ba \text{ for all } a \in A\}.$$

When  $A$  is a C\*-algebra, the relevant notion of ultrapower is the C\*-algebra ultrapower and when  $A$  is a  $\text{II}_1$  factor and the notion of ultrapower is the tracial ultrapower of von Neumann algebras. When  $A$  is a unital C\*-algebra, the central sequence algebra is denoted  $F(A)$  by Kirchberg in [23]

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<sup>1</sup>All ultrafilters are assumed to be nonprincipal ultrafilters on  $\mathbb{N}$  and  $A^{\mathcal{U}}$  denotes the ultrapower of  $A$  associated with  $\mathcal{U}$ .

(suppressing the choice of ultrafilter  $\mathcal{U}$ ). Kirchberg also defines  $F(A)$  for non-unital  $C^*$ -algebras in such a way that Theorem 2.1 extends to the case where  $A$  is non-unital. The relative commutant (and the central sequence algebra in particular) plays an even more significant role in classification of both  $\text{II}_1$  factors and  $C^*$ -algebras than the ultrapower itself ([33], [29], [6], [24], [23], [31], [28], [26]...). Relative commutants do not satisfy the standard form of Łoś' theorem since  $A$  is not even a subalgebra of  $A' \cap A^{\mathcal{U}}$  and are in general only known to be quantifier-free saturated (see §1.1.2). The latter property is closely related to Kirchberg's  $\varepsilon$ -test (see [23, Lemma A.1] or [25, Lemma 3.1]). Moreover, unlike the ultrapower, the construction of a relative commutant does not have an analogue in general model theoretic terms. We aim to show that the central sequence algebra of strongly self-absorbing algebras is no different from its ultrapower.

For a  $C^*$ -algebra  $A$  and a unital  $C^*$ -algebra  $B$ , two  $*$ -homomorphisms  $\Phi$  and  $\Psi$  from  $A$  to  $B$  are *approximately unitarily equivalent* if there is a net of unitaries  $u_\lambda$  in  $B$  such that  $\text{Ad } u_\lambda \circ \Phi$  converges to  $\Psi$  in the point-norm topology. An endomorphism of a  $C^*$ -algebra is *approximately inner* if it is approximately unitarily equivalent to the identity map. Let  $D$  be a separable  $C^*$ -algebra. Recall the following definitions from [9] and [35].

- (1)  $D$  has an *approximately inner flip* if the automorphism of  $D \otimes_{\min} D$  that interchanges  $a \otimes b$  and  $b \otimes a$  is approximately inner.<sup>2</sup>
- (2)  $D$  has an *approximately inner half-flip* if the maps  $\text{id}_D \otimes 1_D$  and  $1_D \otimes \text{id}_D$  from  $D$  into  $D \otimes_{\min} D$  are approximately unitarily equivalent.
- (3) A  $C^*$ -algebra  $A$  is  *$D$ -absorbing* if  $A \otimes_{\min} D \cong A$ .
- (4)  $D$  is *strongly self absorbing* if  $D$  is  $D$ -absorbing and the map  $\text{id}_D \otimes 1_D$  is approximately unitarily equivalent to an isomorphism between  $D$  and  $D \otimes_{\min} D$ .

If  $D$  is strongly self-absorbing then it has an approximately inner flip ([35]), and if  $D$  has an approximately inner flip then it clearly has an approximately inner half-flip. Converses to both of these implications are well-known to be false (see Example 3.6). The weakest of these properties implies that  $D$  is nuclear, simple, unital, and has at most one trace ([9]). If  $D$  has approximately inner half-flip then  $D^{\otimes \infty}$  is strongly self-absorbing ([35, Proposition 1.9]).

We now state Theorem 1 in its weaker form. The result will be strengthened later.

**Theorem 1.** *Assume  $D$  has approximately inner half-flip and  $A$  is a separable,  $D$ -absorbing, unital  $C^*$ -algebra. Then the continuum hypothesis (CH) implies the following.*

- (1) *If  $\Phi: D \rightarrow A^{\mathcal{U}}$  is a unital  $*$ -homomorphism then  $\Phi(D)' \cap A^{\mathcal{U}}$  is isomorphic to  $A^{\mathcal{U}}$ .*

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<sup>2</sup> $\otimes_{\min}$  denotes the minimal, or spatial, tensor product of  $C^*$ -algebras. In most of this paper, we work with tensor products where one factor is nuclear, and then simply write  $\otimes$  (since all  $C^*$ -tensor norms are the same).

- (2) If  $\Phi: D \rightarrow \ell_\infty(A)/c_0(A)$  is a unital \*-homomorphism then  $\Phi(D)' \cap \ell_\infty(A)/c_0(A)$  is isomorphic to  $\ell_\infty(A)/c_0(A)$ .

Moreover, in both cases, the inclusion maps are approximately unitarily equivalent to isomorphisms.

Theorem 1 is a consequence of Corollary 2.13, stating that, assuming CH,  $\ell_\infty(A)/c_{\mathcal{F}}(A)$  and  $A' \cap \ell_\infty(A)/c_{\mathcal{F}}(A)$  are isomorphic for a number of free filters  $\mathcal{F}$ . Our second result, Theorem 2, does not require CH or any other additional set-theoretic assumptions. It is stated in terms of logic of metric structures ([1]) and its adaptation to C\*-algebras ([14]). Some acquaintance with this logic is expected from the reader; §1 contains a brief introduction and all the relevant definitions can be found in [14].

We say that  $C$  is potentially  $D$ -absorbing if  $C$  is elementarily equivalent to a  $D$ -absorbing C\*-algebra (see §2). If  $A$  is potentially  $D$ -absorbing then  $A^{\mathcal{F}}$  is potentially  $D$ -absorbing for every filter  $\mathcal{F}$  (see Lemma 2.6).

**Theorem 2.** *Assume  $C$  and  $D$  are unital C\*-algebras,  $D$  has approximately inner half-flip and  $C$  is countably saturated and potentially  $D$ -absorbing. Then:*

- (1) *All unital \*-homomorphisms of  $D$  into  $C$  are unitarily conjugate.*
- (2) *Fixing an inclusion of  $D$  in  $C$ , we have that  $D' \cap C$  is an elementary submodel of  $C$ , and*
- (3)  *$D' \cap C$  is countably saturated.*

If the continuum hypothesis holds and  $C$  has density character  $\aleph_1$  then items (2) and (3) of Theorem 2 imply  $D' \cap C \cong C$  by a transfinite back-and-forth argument.

The natural intuition that an ultrapower  $A^{\mathcal{U}}$  of a  $D$ -absorbing C\*-algebra  $A$  is  $D$ -absorbing is wrong because countably quantifier-free saturated C\*-algebras are, by [21], tensorially indecomposable. Since ultrapowers of  $D$ -absorbing C\*-algebras are potentially  $D$ -absorbing (Lemma 2.6) Theorem 2 morally justifies this intuition.

*Proof of Theorem 2.* Item (1) was essentially proved in [9], see Lemma 2.8. (2) is Theorem 2.9. The quantifier-free case of (3) is proved in [11] and the general case is Corollary 2.11.  $\square$

The proofs of Theorem 1 and Theorem 2 apply in the case of  $\text{II}_1$  factors where, by a result of Connes, the only strongly self-absorbing example is the hyperfinite  $\text{II}_1$  factor  $R$ . In the following the ultrapower is the tracial von Neumann ultrapower.

**Theorem 3.** *For any McDuff factor  $M$ , given an embedding of  $R$  into  $M^{\mathcal{U}}$ , we have  $R' \cap M^{\mathcal{U}}$  is countably saturated and  $R' \cap M^{\mathcal{U}}$  is an elementary submodel of  $M^{\mathcal{U}}$ . If CH holds then  $R' \cap M^{\mathcal{U}}$  is isomorphic to  $M^{\mathcal{U}}$ .*

*If  $M$  denotes the von Neumann subalgebra of  $R^{\mathcal{U}}$  generated by  $R$  and  $R' \cap R^{\mathcal{U}}$  then  $R' \cap R^{\mathcal{U}}$  is an elementary submodel of  $M$  which itself is an elementary submodel of  $R^{\mathcal{U}}$ .*

A topological dynamical system associated to unitary equivalence classes of  $*$ -homomorphisms from a  $C^*$ -algebra  $A$  into an ultrapower of a  $C^*$ -algebra  $B$  was introduced by Brown in [4]. Much is known about such systems in case of  $\text{II}_1$  factors and some of our results can be recast in Brown's terminology to give information about such dynamical systems when  $A$  is strongly self-absorbing and  $B$  is  $A$ -absorbing.

We also give a new characterization of strongly self-absorbing algebras as those unital  $C^*$ -algebras  $D$  such that all unital  $*$ -homomorphisms of  $D$  into its ultrapower are conjugate and  $D$  is elementarily equivalent to  $D \otimes D$  (Theorem 2.15).

**Organization of the paper.** In §1 we review the basics of continuous model theory and countable saturation. §1.2 is about arbitrary metric structures and in it we prove some continuous variants of the Tarski–Vaught criterion for elementarity. The main results of the paper, including Theorem 2 and a strengthening of Theorem 1, are proved in §2.2. The model-theoretic characterization of strongly self-absorbing algebras mentioned above is given in §2.3. In §2.4 we show that (unlike the class of all separable, nuclear, simple, unital  $C^*$ -algebras) strongly self-absorbing algebras are smoothly classifiable by their universal theories, while  $C^*$ -algebras with approximately inner half-flip are not. Several limiting examples are given in §3.

## 1. PRELIMINARIES

**1.1. Logic of  $C^*$ -algebras.** We quickly review basic notions from the logic of metric structures ([1]) as modified for  $C^*$ -algebras (proofs and more details can be found in [14] or [12]).

**1.1.1. Formulas, theories, elementary submodels.** The *formulas* for  $C^*$ -algebras are recursively defined as follows.

- (1) Terms are  $*$ -polynomials with complex coefficients in noncommuting variables  $x_n$ , for  $n \geq 0$ .
- (2) Atomic formulas are expressions of the form  $\|P\|$  where  $P$  is a term.
- (3) Formulas form the smallest set that includes all atomic formulas and satisfies the following requirements.
  - (a) If  $n \geq 1$ ,  $\phi_j$  for  $0 \leq j < n$  are formulas, and  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous, then  $f(\phi_0, \phi_1, \dots, \phi_{n-1})$  is a formula.
  - (b) If  $\phi$  is a formula and  $x$  is  $x_j$  for some  $j \geq 0$ , then  $\sup_{\|x\| \leq 1} \phi$  and  $\inf_{\|x\| \leq 1} \phi$  are formulas.

Formulas in the set containing all atomic formulas and closed only under condition 3(a) are called *quantifier-free formulas*.

This definition of formula is slightly more restrictive than the one given in [14] in that we consider only the unit ball of the  $C^*$ -algebra as a domain of quantification. This simplification does not alter the expressive power of the logic and it suffices for our present purposes.

We will frequently use  $\bar{x}$  to denote the  $n$ -tuple  $(x_0, \dots, x_{n-1})$  or  $\omega$ -tuple  $(x_0, x_1, \dots)$  of variables, and likewise  $\bar{a}$  (or  $\bar{b}$ , etc.) to denote a finite or countable tuple of elements in a C\*-algebra  $A$ . We will write  $\bar{a} \in A$  to mean that  $a_j \in A$  for each  $a_j$  in the tuple  $\bar{a}$ . A formula with variables included in the tuple  $\bar{x}$  will be denoted  $\phi(\bar{x})$  (it is convenient to sometimes allow  $\bar{x}$  to be a countable tuple, even though the definition of a formula only allows finitely many of the variables to be used).

Let  $n = 0, 1, \dots, \omega$ . Given a formula  $\phi(\bar{x})$ , where  $\bar{x}$  is an  $n$ -tuple, a C\*-algebra  $A$  and an  $n$ -tuple  $\bar{a}$  in  $A$ , one can define the interpretation of  $\phi(\bar{a})$  in  $A$  recursively, by following the structure of  $\phi$  as in (1)–(3). This interpretation, denoted by  $\phi(\bar{a})^A$ , defines a function  $\phi^A: A^n \rightarrow \mathbb{R}$ . This function is uniformly continuous on the set of  $n$ -tuples in the unit ball of  $A$ , and the modulus of uniform continuity depends only on  $\phi$  and not on  $A$ . We denote the set of formulas whose free variables are included in  $\bar{x}$ , by  $\mathbb{F}^{\bar{x}}$ .

We can expand the notion of formula slightly by allowing parameters. If  $A \subseteq B$ ,  $\varphi(\bar{x}, \bar{y})$  is a formula and  $\bar{a} \in A$  has the same length as  $\bar{y}$  then we call  $\varphi(\bar{x}, \bar{a})$  a formula with parameters in  $A$ . We denote by  $\mathbb{F}_A^{\bar{x}}$  the set of formulas with free variables included in  $\bar{x}$  and parameters in  $A$ .

A formula is a *sentence* if it has no free variables; we denote the set of sentences as  $\mathbb{F}^0$ . For a sentence  $\phi$  the interpretation  $\phi^A$  is a real number. Note that  $\mathbb{F}^0$  is an  $\mathbb{R}$ -algebra and that for every  $A$  the map  $\phi \mapsto \phi^A$  is linear.

The *theory* of a C\*-algebra  $A$  is the kernel of this map on  $\mathbb{F}^0$ ,

$$\text{Th}(A) := \{\phi \in \mathbb{F}^0 : \phi^A = 0\}.$$

Since  $\phi^A = r$  if and only if  $|\phi - r| \in \text{Th}(A)$ , the theory of  $A$  can be identified with the linear functional  $\phi \mapsto \phi^A$ . We say that  $A$  and  $B$  are *elementarily equivalent*, in symbols  $A \equiv B$ , if  $\text{Th}(A) = \text{Th}(B)$ .

If  $A$  is a subalgebra of  $B$  and for every  $n$ -ary formula  $\phi$  the interpretations  $\phi^A$  and  $\phi^B$  agree on  $A^n$  then we write  $A \prec B$  and say that  $A$  is an *elementary submodel* of  $B$  and that  $B$  is an *elementary extension* of  $A$ . The following easily checked facts will be used tacitly. If  $A \subseteq B \subseteq C$  then  $A \prec B$  and  $B \prec C$  implies  $A \prec C$ . Also  $A \prec C$  and  $B \prec C$  implies  $A \prec B$ . (Note, however, that  $A \prec C$  and  $A \prec B$  does not imply  $B \prec C$ .) By Łoś' theorem an ultrapower is always an elementary extension of  $A$ ,  $A \prec A^{\mathcal{U}}$ , via the diagonal embedding.

**1.1.2. Types and saturation.** If  $A \subseteq B \prec C$  and  $\bar{c} \in C$  then the complete type of  $\bar{c}$  over  $A$  with respect to  $\text{Th}(B)$  is the linear functional  $p: \mathbb{F}_A^{\bar{x}} \rightarrow \mathbb{R}$  given by  $p(\varphi(\bar{x}, \bar{a})) = \varphi^C(\bar{c}, \bar{a})$ . A *type* over  $A$  with respect to  $\text{Th}(B)$  is the restriction of a complete type to a subset of  $\mathbb{F}_A^{\bar{x}}$ ; if the subset contains only quantifier-free formulas then we say that the type is a quantifier-free type. As a type can be extended by linearity to a subspace of  $\mathbb{F}_A^{\bar{x}}$ , it is determined by its kernel. For  $\varphi \in \mathbb{F}_A^{\bar{x}}$  and a type  $p$ , we write  $\varphi \in p$  if  $p(\varphi) = 0$ . A set of formulas  $\Sigma$  with parameters in  $A$  is *consistent* with  $\text{Th}(B)$  if for some type

$p$  over  $A$  with respect to  $\text{Th}(B)$ , we have  $\Sigma \subseteq p$ . The following result, which is used throughout, can be proved by an application of Łoś' theorem.

**Proposition 1.1.** *Suppose that  $A \subseteq B$ , both metric structures in the same language, and  $\Sigma$  is a set of formulas with parameters in  $A$ . Then  $\Sigma$  is consistent with  $\text{Th}(B)$  if and only if every finite subset  $\Sigma_0$  of  $\Sigma$  and  $\varepsilon > 0$ ,  $\Sigma_0$  can be  $\varepsilon$ -approximated in  $B$  i.e., there exists  $\bar{b} \in B$  such that  $|\varphi^B(\bar{b}, \bar{a})| < \varepsilon$  for every  $\varphi(\bar{x}, \bar{a}) \in \Sigma_0$ .*

*If  $A$  is separable then every type over  $A$  is realized in  $A^{\mathcal{U}}$ .*

A  $C^*$ -algebra  $C$  is *countably saturated* if for every separable subset  $A$  of  $C$ , every type over  $A$  is realized in  $C$ ; it is countably quantifier-free saturated if over every separable subset  $C$ , every quantifier-free type is realized in  $C$ . The second key property of ultrapowers alluded to in the introduction is the fact that if  $\mathcal{U}$  is a nonprincipal ultrafilter on  $\mathbb{N}$  then  $B^{\mathcal{U}}$  is countably saturated for every  $C^*$ -algebra (and even every metric structure)  $B$ . We record the following classical result which is proved by a transfinite back-and-forth construction. The density character of a metric space is the minimal cardinality of a dense subset.

**Theorem 1.2.** *Assume  $C$  and  $D$  are elementarily equivalent, countably saturated, and both have density character  $\aleph_1$ . Then  $C$  and  $D$  are isomorphic.*

*In particular, if the continuum hypothesis holds and  $A$  is separable then every countably saturated model of density character  $\aleph_1$  elementarily equivalent to  $A$  is isomorphic to every ultrapower of  $A$  associated to a nonprincipal ultrafilter on  $\mathbb{N}$ .  $\square$*

We should note that, unless the continuum hypothesis holds, there are no infinite-dimensional and countably saturated  $C^*$ -algebras of density character  $\aleph_1$  by the main result in [14].

We end this section with a well-known lemma.

**Lemma 1.3.** *For any pair of  $C^*$ -algebras  $A, B$ , the set of elementary embeddings from  $A$  to  $B$  is closed in the point-norm topology. In particular, if  $\Phi: A \rightarrow B$  is approximately unitarily equivalent to an elementary embedding  $\Psi: A \rightarrow B$  then  $\Phi$  is an elementary embedding.*

*Proof.* Let  $\Phi_\lambda$ , for  $\lambda \in \Lambda$ , be a net of elementary maps from  $A$  to  $B$  that converges to  $\Phi$  point-norm topology. In order to check that  $\Phi$  is elementary fix a formula  $\psi(\bar{x})$ . Then for every  $\bar{a}$  in  $A$  of the same length as  $\bar{x}$  we have that  $\psi(\bar{a})^A = \psi(\Phi_\lambda(\bar{a}))^B$  for all  $\lambda$ . If  $\bar{b} = \lim_\lambda \Phi_\lambda(\bar{a})$  then by the continuity of the interpretation of  $\psi$  we have  $\psi(\bar{b})^B = \lim_\lambda \psi(\Phi_\lambda(\bar{a}))^B = \psi(\bar{a})^A$ . Since  $\psi$  was arbitrary this completes the proof. The second statement follows immediately.  $\square$

**1.2. Criteria for elementarity.** Lemmas in this subsection are stated and proved for general metric structures (as in [1]), although we shall need only the case when they are  $C^*$ -algebras. Although not technically difficult, these

lemmas do not appear elsewhere to the best of our knowledge. They were inspired by analysis of the model-theoretic behaviour of central sequences of strongly self-absorbing algebras. In all of these lemmas  $A, B$  and  $C$  denote metric structures in the same language. The notion of elementary substructure is a general one for metric structures and we prove these results in that generality. For the reader interested only in C\*-algebras, this notion agrees with the earlier notation and these results can be read as statements about C\*-algebras without affecting the rest of the paper.

**Lemma 1.4.** *Assume  $C \prec B$  and  $C \subseteq A \subseteq B$ . Assume moreover that for every  $m$  and every  $\bar{a} \in A$  we have*

$$\inf\{\text{dist}(\alpha(\bar{a}), C^m) : \alpha \in \text{Aut}(B), \alpha \upharpoonright A \in \text{Aut}(A)\} = 0.$$

*Then  $C \prec A \prec B$ .*

*Proof.* It suffices to prove that  $A \prec B$ , since from this it follows that  $C \prec A$ . Fix a formula  $\inf_y \phi(\bar{x}, y)$  and  $\bar{a}$  in  $A$ . By the Tarski–Vaught test ([1, Proposition 4.5]) we need to check that  $\inf_{y \in A} \phi(\bar{a}, y)^B = \inf_{y \in B} \phi(\bar{a}, y)^B$ .

Since  $C \prec B$ , if  $\bar{c}$  is in  $C^m$  we have

$$\inf_{y \in B} \phi(\bar{c}, y)^B = \inf_{y \in C} \phi(\bar{c}, y)^B$$

for all  $m$ . The monotonicity of taking inf implies

$$\inf_{y \in B} \phi(\bar{c}, y)^B \leq \inf_{y \in A} \phi(\bar{c}, y)^B \leq \inf_{y \in C} \phi(\bar{c}, y)^B$$

and therefore we have the equality

$$\inf_{y \in A} \phi(\bar{c}, y)^B = \inf_{y \in B} \phi(\bar{c}, y)^B.$$

Fix a sequence  $\alpha_n$ , for  $n \in \mathbb{N}$ , of automorphisms of  $B$  such that  $\alpha_n \upharpoonright A$  is an automorphism of  $A$  such that  $\lim_n \text{dist}(\alpha_n(\bar{a}), C^m) = 0$ . Let  $\bar{a}(n)$  be a tuple from  $C$  such that  $\lim_n \text{dist}(\alpha_n(\bar{a}), \bar{a}(n)) = 0$ .

Then  $\inf_{y \in B} \phi(\bar{a}, y)^B = \inf_{y \in B} \phi(\alpha_n(\bar{a}), y)^B$  and  $\lim_n |\inf_{y \in B} \phi(\alpha_n(\bar{a}))^B - \inf_{y \in B} \phi(\bar{a}(n))^B| = 0$  by the uniform continuity of the interpretation of  $\phi$ . By the above, the conclusion follows.  $\square$

**Lemma 1.5.** *Assume  $A \subseteq B$ . Assume in addition that for all  $m, n$ , and  $\bar{b} \in B^m$  and  $\bar{a} \in A^n$  we have*

$$\inf\{d(\alpha(\bar{a}), \bar{a}) + \text{dist}(\alpha(\bar{b}), A^m) : \alpha \in \text{Aut}(B)\} = 0.$$

*Then  $A \prec B$ . If  $B$  is in addition separable, then  $A \cong B$ .*

*Proof.* We need to check the Tarski–Vaught test, that for every  $n$ -ary formula  $\inf_y \psi(\bar{x}, y)$  and all  $\bar{a} \in A$  we have

$$\inf_{y \in B} \psi(\bar{a}, y)^B = \inf_{y \in A} \psi(\bar{a}, y)^B.$$

We clearly have “ $\leq$ .” In order to prove “ $\geq$ ,” set  $r := \inf_{y \in B} \psi(\bar{a}, y)^B$ . Fix  $\varepsilon > 0$  and  $b \in B$  such that  $\psi(\bar{a}, b)^B < r + \varepsilon/2$ . Now  $\psi^B$  is uniformly



continuous so fix  $\delta$  corresponding to  $\varepsilon/2$  for the uniform continuity modulus of  $\psi$ . Pick an automorphism  $\alpha$  of  $B$  such that

$$d(\bar{a}, \alpha(\bar{a})) + d(\alpha(b), A) < \delta.$$

Then  $\psi(\alpha(\bar{a}), \alpha(b))^B = \psi(\bar{a}, b)^B < r + \varepsilon/2$ . Now pick  $b' \in A$  such that  $d(\alpha(b), b') < \delta$ . Since we also have  $d(\bar{a}, \alpha(\bar{a})) < \delta$ , altogether we get

$$|\psi^B(\bar{a}, b') - \psi^B(\alpha(\bar{a}), \alpha(b))| < \varepsilon/2$$

Since  $b' \in A$ , the conclusion follows.

In the case when  $B$  is separable, this is a standard back and forth argument.  $\square$

We record a slightly simpler form of Lemma 1.5 (for use when  $B$  is countably quantifier-free saturated, under which condition this version is equivalent).

**Lemma 1.6.** *Assume  $A \subseteq B$  and that for all  $m, n$  and  $\bar{b} \in B^m$  and  $\bar{a} \in A^n$  there exists an automorphism  $\alpha$  of  $B$  such that (with a slight abuse of notation)  $\alpha(\bar{a}) = \bar{a}$  and  $\alpha(\bar{b}) \in A$ . Then  $A \prec B$ .  $\square$*

## 2. MAIN RESULTS

Let  $\mathcal{F}$  be a filter on  $\mathbb{N}$  (we stick to  $\mathbb{N}$  only for convenience). Identifying  $a \in \ell_\infty(A)$  with the sequence  $(a(n) : n \in \mathbb{N})$  we set

$$c_{\mathcal{F}}(A) := \{a \in \ell_\infty(A) : \inf_{X \in \mathcal{F}} \sup_{n \in X} \|a(n)\| = 0\}.$$

This is a two-sided, norm-closed, (therefore) self-adjoint ideal in  $\ell_\infty(A)$ . Set

$$A^{\mathcal{F}} := \ell_\infty(A)/c_{\mathcal{F}}(A).$$

Important special cases are ultrapowers (when  $\mathcal{F}$  is an ultrafilter) and the asymptotic sequence algebra (when  $\mathcal{F}$  is the Frechét filter).

We record a key fact about C\*-algebras with approximately inner half-flip taken from [33, Theorem 7.2.2] (as extracted from [9]).

**Theorem 2.1.** *Suppose  $A$  and  $D$  are separable, unital C\*-algebras,  $D$  has approximately inner half-flip, and  $\mathcal{F}$  is a free filter (i.e., it includes the Frechét filter).*

- (1) *Then  $A$  is  $D$ -absorbing if  $D$  unitaly embeds into  $A' \cap A^{\mathcal{F}}$ .*
- (2) *If  $D$  is in addition strongly self-absorbing then the converse holds, so  $A$  is  $D$ -absorbing if and only if  $D$  unitaly embeds into  $A' \cap A^{\mathcal{F}}$ .*

*Proof.* Only the converse direction in (2) may require a proof. Suppose  $D$  is strongly self-absorbing and  $A$  is  $D$ -absorbing. Then  $D$  unitaly embeds into  $A' \cap A^{\mathcal{F}_0}$ , where  $\mathcal{F}_0$  is the Frechét filter. So  $A^{\mathcal{F}}$  is a quotient of  $A^{\mathcal{F}_0}$  and the quotient map sends  $A' \cap A^{\mathcal{F}}$  into  $A' \cap A^{\mathcal{F}_0}$ . Since  $D$  is a simple unital subalgebra of  $A' \cap A^{\mathcal{F}_0}$ , the quotient map is injective on  $D$ .  $\square$

The fact that  $D$  unitaly embeds into  $A' \cap A^{\mathcal{F}_0}$  when  $A$  is  $D$ -absorbing easily follows from the fact that  $A' \cap A^{\mathcal{F}_0}$  is countably quantifier-free saturated and that  $A \models \mathbb{T}_D$  (see §2.1). Note that  $A' \cap A^{\mathcal{F}}$  need not be countably saturated (e.g. if  $\mathcal{F} = \mathcal{Z}_0$ , by an argument from Example 3.2).

The choice of the ultrafilter  $\mathcal{U}$  is irrelevant in this statement, although the isomorphism type of  $A' \cap A^{\mathcal{U}}$  depends on the choice of  $\mathcal{U}$  unless the continuum hypothesis holds; see [13].

Example 3.10 shows that the converse in Theorem 2.1 need not hold if  $D$  has approximately inner half-flip, or even approximately inner flip, but isn't strongly self-absorbing.

**2.1. Theory  $\mathbb{T}_D$ .** Fix a separable C\*-algebra  $D$ . Suppose that  $p_1, \dots, p_n$  are \*-polynomials with complex coefficients in the noncommuting variables  $y_1, \dots, y_n$  and let  $d_1, \dots, d_n \in D$ . Let  $r_j = \|p_j(d_1, \dots, d_n)\|$  for  $j = 1, \dots, n$ . Let  $\Delta = (p_1, \dots, p_n, d_1, \dots, d_n)$ . We define the formulas

$$\begin{aligned} \phi_{D,\Delta}(\bar{x}, \bar{y}) &:= \max_{i < n, j < n} \|[x_i, y_j]\| + \max_{j < n} |r_j - \|p_j(\bar{y})\||, \\ \psi_{D,\Delta} &:= \sup_{\bar{x}} \inf_{\bar{y}} \phi_{D,\Delta}(\bar{x}, \bar{y}). \end{aligned}$$

Set

$$\mathbb{T}_D := \{\psi_{D,\Delta} : \Delta\},$$

where  $\Delta$  ranges over all  $(p_1, \dots, p_n, d_1, \dots, d_n)$ , over all  $n \in \mathbb{N}$ . Recall that  $\text{Th}(A) = \{\phi : \phi^A = 0\}$ . We write  $A \models \mathbb{T}$  if  $\mathbb{T} \subseteq \text{Th}(A)$ . The C\*-algebra  $D$  in the following lemma is not assumed to have any special properties such as approximately inner half-flip and in particular need not be unital.

**Lemma 2.2.** *For separable C\*-algebras  $A$  and  $D$ , and an ultrafilter  $\mathcal{U}$ ,  $D$  embeds into the central sequence algebra  $A' \cap A^{\mathcal{U}}$  if and only if  $A \models \mathbb{T}_D$ .*

*Proof.* Assume  $D$  embeds into  $A' \cap A^{\mathcal{U}}$  and fix  $n$ . Since  $A$  is an elementary submodel of  $A^{\mathcal{U}}$ , for every  $n$ -tuple  $\bar{a}$  in  $A$  we have  $\phi_{D,n}(\bar{a})^A = \phi_{D,n}(\bar{a})^{A^{\mathcal{U}}} = 0$ . Therefore  $A \models \mathbb{T}_D$ .

Now assume  $A \models \mathbb{T}_D$ . Introduce variables  $y_d$  for every  $d \in X$  where  $X$  is a countable dense subset of  $D$ . Consider the type over  $A$  given by all the formulas  $\phi_{D,\Delta}(a_1, \dots, a_n, y_{d_1}, \dots, y_{d_n})$  where  $\Delta = (p_1, \dots, p_n, d_1, \dots, d_n)$  and  $a_1, \dots, a_n \in A$ . This type is consistent because  $A \models \mathbb{T}_D$  and hence is realized in  $A^{\mathcal{U}}$  by some set  $\{c_d : d \in X\}$ . The map  $d \mapsto c_d$  can be extended to a \*-homomorphism from  $D$  into  $A' \cap A^{\mathcal{U}}$ .  $\square$

Theorem 2.1 and Lemma 2.2 imply the following.

**Lemma 2.3.** *If  $A$  is separable and  $D$  has approximately inner half-flip then  $A \models \mathbb{T}_D$  implies that  $A$  is  $D$ -absorbing. If  $D$  is in addition strongly self-absorbing then  $A \models \mathbb{T}_D$  if and only if  $A$  is  $D$ -absorbing.*  $\square$

The separability of  $A$  is necessary in Lemma 2.3 since by [21] ultrapowers are always tensorially indecomposable. By Example 3.10  $A$  being  $D$ -absorbing for some  $D$  with approximately inner flip does not imply  $A \models \mathbb{T}_D$ .

**Definition 2.4.** Fix a separable, unital  $C^*$ -algebra  $D$  and a (typically non-separable)  $C^*$ -algebra  $C$ .

- (1) We say  $C$  is *potentially  $D$ -absorbing* if  $C \models \mathbb{T}_D$ .
- (2) We say  $C$  is  *$D$ -saturated* if for every separable  $X \subseteq C$  there is a unital  $*$ -homomorphism of  $D$  into  $X' \cap C$ .

The notion that a  $C^*$ -algebra is  $D$ -saturated is meant to convey the idea that  $D$  is present everywhere in  $C$ , so that  $C$  is saturated with isomorphic copies of  $D$ . If  $C$  is  $D$ -saturated then a copy of  $D$  can be found in the relative commutant of any separable subalgebra of  $C$ .

Notice that if a  $C^*$ -algebra is potentially  $D$ -absorbing and countably saturated then it is  $D$ -saturated. In fact,

**Lemma 2.5.** *Suppose  $C$  and  $D$  are unital  $C^*$ -algebras.*

- (1) *If  $C$  is countably saturated then it is potentially  $D$ -absorbing if and only if it is  $D$ -saturated.*
- (2) *If  $D$  has approximately inner half-flip and  $C$  is potentially  $D$ -absorbing then every separable  $A$  elementarily equivalent to  $C$  is  $D$ -absorbing.*
- (3) *If  $D$  is strongly self-absorbing then  $C$  is potentially  $D$ -absorbing if and only if every separable  $A$  elementarily equivalent to  $C$  is  $D$ -absorbing.  $\square$*

Although Łoś' theorem fails for the embedding of  $A$  into  $A^{\mathcal{F}}$  unless  $\mathcal{F}$  is an ultrafilter, the theory of  $A^{\mathcal{F}}$  can be computed from the theory of  $A$  and some information on  $\mathcal{F}$  by Ghasemi's metric version of the Feferman–Vaught theorem ([20]). The following lemma is an easy instance of this theorem.

**Lemma 2.6.** *Assume  $\mathcal{F}$  is a filter on  $\mathbb{N}$  and  $B$  is separable. Then  $A$  is potentially  $B$ -absorbing if and only if  $A^{\mathcal{F}}$  is potentially  $B$ -absorbing.*

*Proof.* We shall prove that  $(\psi_{B,\Delta})^{A^{\mathcal{F}}} = 0$  if and only if  $(\psi_{B,\Delta})^A = 0$  for all  $\Delta$ . Fix  $\Delta = (p_1, \dots, p_m, d_1, \dots, d_m)$  and  $m$ -tuples  $\bar{a}$  and  $\bar{b}$  in  $A$ , with representing sequences  $\langle \bar{a}(n) : n \in \mathbb{N} \rangle$  and  $\langle \bar{b}(n) : n \in \mathbb{N} \rangle$ , respectively. Then

$$\begin{aligned} \phi_{D,\Delta}(\bar{a}, \bar{b})^{A^{\mathcal{F}}} &= \max_{i < m, j < m} \|[a_i, b_j]\|^{A^{\mathcal{F}}} + \max_{j < m} |r_j - \|p_j(\bar{b})\|^{A^{\mathcal{F}}}| \\ &= \max_{i < m, j < m} \inf_{X \in \mathcal{F}} \sup_{n \in X} \|[a_i(n), b_j(n)]\| + \max_{j < m} \inf_{X \in \mathcal{F}} \sup_{n \in X} |r_j - \|p_j(\bar{b}(n))\|| \end{aligned}$$

and

$$\inf_{X \in \mathcal{F}} \sup_{n \in X} \phi_{D,\Delta}(\bar{a}(n), \bar{b}(n)) \leq \phi_{D,\Delta}(\bar{a}, \bar{b})^{A^{lc\mathcal{F}}} \leq 2n \inf_{X \in \mathcal{F}} \sup_{n \in X} \phi_{D,\Delta}(\bar{a}(n), \bar{b}(n)).$$

Therefore  $\psi_{D,\Delta} := \sup_{\bar{x}} \inf_{\bar{y}} \phi_{D,\Delta}(\bar{x}, \bar{y})$  satisfies

$$(\psi_{D,\Delta})^A \leq (\psi_{D,\Delta})^{A^{\mathcal{F}}} \leq 2n(\psi_{D,\Delta})^A,$$

and the conclusion follows.  $\square$

This result also applies to asymptotic sequence algebras which were proved to be countably saturated in [16, Theorem 1.5].

**Proposition 2.7.** *Assume  $A$  and  $B$  are separable, unital C\*-algebras. Then for a nonprincipal filter  $\mathcal{F}$  on  $\mathbb{N}$  such that  $A^{\mathcal{F}}$  is countably saturated, the following are equivalent:*

- (1)  $A$  is potentially  $B$ -absorbing.
- (2)  $A' \cap A^{\mathcal{F}}$  is potentially  $B$ -absorbing.
- (3)  $B$  embeds into  $A' \cap A^{\mathcal{F}}$ .

*Proof.* Lemma 2.6 and Lemma 2.2 together imply this result. □

A few remarks on the optimality of Proposition 2.7 are in order.

(a) There are separable  $A$  and  $B$  such that  $A \models \mathbb{T}_B$  but  $A \otimes B$  is not isomorphic to  $A$ . If  $A$  is an infinite-dimensional unital C\*-algebra, then  $A' \cap A^{\mathcal{U}}$  is nonseparable and in particular  $C([0, 1])$  embeds into it. However, if the center of  $A$  is trivial then  $A$  does not absorb  $C([0, 1])$  tensorially.

(b) There are separable  $A$  and  $B$  such that  $B$  has approximately inner flip and  $A \otimes B \cong A$ , but  $A \not\models \mathbb{T}_B$  (see Example 3.10). However, if  $A \cong A \otimes B$  and the isomorphism is approximately unitarily equivalent to the inclusion map on  $A$  (approximate unitary equivalence is typically determined by  $K$ -theoretic invariants; see Theorem 3.3 and [27]), then  $A \otimes 1$  is an elementary submodel of  $A \otimes B$  and therefore  $A \models \mathbb{T}_B$ .

(c) If  $A \models \mathbb{T}_B$  and  $B$  is nuclear and separable then  $A \models \mathbb{T}_{B^{\otimes \infty}}$  (the nuclearity of  $B$  is used only to assure that  $B^{\otimes \infty}$  is uniquely defined). In particular, if  $B$  is nuclear then every  $B$ -saturated C\*-algebra is  $B^{\otimes \infty}$ -saturated. This was proved by Kirchberg in [23] for  $A' \cap A^{\mathcal{U}}$ .

(d) There exist nonprincipal filters  $\mathcal{F}$  such that  $A^{\mathcal{F}}$  is not necessarily countably saturated (Example 3.2). However, many nonprincipal filters  $\mathcal{F}$  on  $\mathbb{N}$  have the property that  $A^{\mathcal{F}}$  is countably saturated for every C\*-algebra  $A$  [16, Theorem 2.7].

The following is essentially contained in [9].

**Lemma 2.8.** *Assume  $C$  and  $D$  are unital C\*-algebras,  $D$  has approximately inner half-flip and  $C$  is  $D$ -saturated. Then all unital \*-homomorphisms of  $D$  into  $C$  are approximately unitarily equivalent. If  $C$  is in addition countably quantifier-free saturated, then all unital \*-homomorphisms from  $D$  to  $C$  are in fact unitarily conjugate.*

*Proof.* This is almost tautological. We include a proof for the reader's convenience. Let  $\Phi_1$  and  $\Phi_2$  be unital \*-homomorphisms of  $D$  into  $C$ . By hypothesis, we can find a unital \*-homomorphism  $\Psi$  from  $D$  into  $C$  whose range commutes with ranges of  $\Phi_1$  and  $\Phi_2$ . It suffices to prove that  $\Psi$  and  $\Phi_i$  are unitarily conjugate, i.e., to prove the assertion in case when  $\Phi_1$  and  $\Phi_2$  have commuting ranges.

In this case,  $d_1 \otimes d_2 \mapsto \Phi_1(d_1)\Phi_2(d_2)$  defines a map  $\Theta : D \otimes D \rightarrow C$ . Let  $u_n$ , for  $n \in \mathbb{N}$ , be a sequence of unitaries in  $D \otimes D$  such that  $\lim_n \text{Ad}(u_n)(d \otimes 1) = 1 \otimes d$ . Then  $\Theta(u_n)$ , for  $n \in \mathbb{N}$ , is a sequence of unitaries in  $C$  which satisfy

$$\lim_n \text{Ad}(\Theta(u_n))\Phi_1(d) = \Phi_2(d),$$

as required.

Now assume that  $C$  is countably quantifier-free saturated. Consider the type  $\mathbf{t}(x)$  consisting of the all the quantifier-free conditions  $\|xx^* - 1\| = 0$ ,  $\|x^*x - 1\| = 0$ ,  $\|x\Phi_2(d)x^* - \Phi_1(d)\| = 0$  for  $d \in D$ . Since  $\Phi_1, \Phi_2$  are approximately unitarily equivalent, any finite subset of this type is approximately realized, so this type is consistent with  $\text{Th}(C)$ . Therefore, since  $C$  is countably quantifier-free saturated, the type is realized, i.e., there is a unitary  $u \in C$  such that  $\text{Ad}(u) \circ \Phi_2 = \Phi_1$ , as required.  $\square$

**2.2. Proofs of Theorem 1 and Theorem 2.** The next few results complete the proof of Theorem 2.

**Theorem 2.9.** *Assume  $C$  is potentially  $D$ -absorbing and countably saturated and  $D$  has approximately inner half-flip. Then, fixing an embedding of  $D$  in  $C$ , we have*

$$D' \cap C \prec C^*(D, D' \cap C) \text{ and } C^*(D, D' \cap C) \prec C.$$

*Proof.* Set  $A := D' \cap C$ . In order to prove  $A \prec C$  we verify the assumptions of Lemma 1.6. Fix  $\bar{a} \in D' \cap C$  and  $\bar{b} \in C$ . Since  $C$  is potentially  $D$ -absorbing we can fix a unital subalgebra  $D_1$  of  $C^*(D, \bar{a}, \bar{b})' \cap C$  isomorphic to  $D$ . Since  $\{\bar{a}\}' \cap C$  is  $D$ -saturated and countably quantifier-free saturated, by Lemma 2.8 there exists a unitary  $u \in \{\bar{a}\}' \cap C$  such that  $D_1 = \text{Ad}(u)(D)$ . Set  $\alpha := \text{Ad}(u) \in \text{Aut}(C)$ , so that  $\alpha(\bar{a}) = \bar{a}$ . By Lemma 1.6 the conclusion follows.

We now prove  $C^*(D, A) \prec C$ . Let us write  $B := C^*(D, A)$ . Since  $A \prec C$ , by Lemma 1.4 we need to show that for every  $m$  and every  $\bar{a} \in B$  we have

$$\inf\{\text{dist}(\alpha(\bar{a}), A^m) : \alpha \in \text{Aut}(C), \alpha \upharpoonright B \in \text{Aut}(B)\} = 0.$$

Since  $B$  is generated by  $D$  and  $A$ , it suffices to consider the case when  $\bar{a} = (d_1, \dots, d_k, c_1, \dots, c_\ell)$  where  $d_i \in D$  and  $c_i \in A$ . Since  $C$  is  $D$ -saturated, so is  $A$ . Therefore there is a unital copy  $D_1$  of  $D$  in  $A$  that commutes with all  $c_i$ . Since  $D$  has approximately inner half-flip, the flip automorphism of  $C^*(D, D_1) \cong D \otimes D_1$  is approximately inner, and the unitaries witnessing this belong to  $C^*(D, D_1) \subseteq B$ . Therefore for any  $\varepsilon > 0$  we can find an inner automorphism of  $B$  that moves all  $d_i$  to within  $\varepsilon$  of  $D_1$  and that does not move any of the  $c_i$ . This automorphism extends to an inner automorphism of  $C$  as required.

Since  $A \prec C$  and  $A \subseteq C^*(D, A) \prec C$ ,  $A \prec C^*(D, A)$  follows.  $\square$

By [11] if  $B$  is countably saturated and  $A \subseteq B$  is separable then  $A' \cap B$  is countably quantifier-free saturated. In general, we don't expect that  $A' \cap B$  is countably saturated, though we have the following.

**Lemma 2.10.** *Assume  $D$  is a separable  $C^*$ -algebra,  $D \subseteq B$ , such that  $B$  is countably saturated and  $D' \cap B \prec B$ . Then  $D' \cap B$  is countably saturated.*

*Proof.* Let  $p$  be a type consistent with  $\text{Th}(D' \cap B)$  over a separable subalgebra  $A \subseteq D' \cap B$  and let  $\bar{x}$  be the tuple of variables occurring in this type.

Consider the set of formulas  $\Sigma$  with parameters in  $A \cup D$ , consisting of all  $\phi(\bar{x}, \bar{a}) \in p$  together with the formula  $\|[x_i, d]\|$  for each  $i$  and each  $d \in D$ . Let us show that  $\Sigma$  is consistent with  $\text{Th}(B)$ .

For this, let  $\Sigma_0$  be a finite subset of  $\Sigma$  and let  $\varepsilon > 0$ . Then, since  $p$  is consistent, all formulas in  $\Sigma$  that are from  $p$  are  $\varepsilon$ -approximated in  $D' \cap B$  by some  $\bar{b} \in D' \cap B$ . Since  $D' \cap B \prec B$ , these conditions are  $\varepsilon$ -approximately realized by  $\bar{b}$  in  $B$ . For the formulas of the form  $\|[x_i, d]\|$ , we also have  $\|[b_i, d]\| = 0$  since  $\bar{b} \in D'$ . This concludes the proof that  $\Sigma$  is consistent with  $\text{Th}(B)$ , i.e., it is a type in  $\text{Th}(B)$ .

Since  $B$  is countably saturated,  $\Sigma$  is realized by some  $\bar{b}$  in  $B$ . The definition of  $\Sigma$  ensures that  $\bar{b}$  in  $D' \cap B$  and that  $\bar{b}$  realizes  $p$  in  $B$ . Finally, since  $D' \cap B \prec B$ , it follows that  $p$  is realized in  $D' \cap B$  by  $\bar{b}$ .  $\square$

Model-theorists will notice that the proof of Lemma 2.10 gives a more general statement. If  $B$  is a countably saturated model and  $p$  is a 1-type over a separable substructure of  $B$  such that the set  $C$  of realizations of  $p$  is an elementary submodel of  $B$ , then  $C$  is countably saturated.

**Corollary 2.11.** *Assume  $D$  has approximately inner half-flip and  $B$  is countably saturated and potentially  $D$ -absorbing. Then  $D' \cap B$  is countably saturated.*

*Proof.* This is a consequence of Lemma 2.10 and Theorem 2.9.  $\square$

*Proof of Theorem 2.* (1) is by Lemma 2.8, (2) is by Theorem 2.9, and (3) is Corollary 2.11.  $\square$

We now turn to the proof of Theorem 1 – in fact, a strengthening of it. We use a nonseparable variant on an intertwining argument.

**Theorem 2.12** (cf. [33, Proposition 2.3.5]). *Suppose  $A \subseteq B$  are unital C\*-algebras with the same unit and that for every separable set  $S \subseteq A$  and every  $b \in B$  there exists a unitary  $u \in S' \cap B$  such that  $ubu^* \in A$ . Then*

- (1)  $A \prec B$ ,
- (2) if  $B$  is countably saturated then so is  $A$ , and
- (3) if  $B$  is countably saturated and of density character  $\aleph_1$  then  $A$  is isomorphic to  $B$  and the inclusion  $A \rightarrow B$  is approximately unitarily equivalent to an isomorphism.

Any infinite-dimensional C\*-algebra is unstable by the main result of [14] which implies that any countably saturated C\*-algebra is of density character continuum. So it follows that the assumptions of (3) above imply the continuum hypothesis.

*Proof.* (1) follows from Lemma 1.6. Once we have (1), then, since  $B$  is countably saturated, any type over a separable submodel  $C$  of  $A$  is realized in  $B$  by some  $b$ . But then by assumption, there is a unitary  $u \in C' \cap B$  such that  $ubu^* \in A$  and so the type is realized in  $A$  as well. In (3), the fact

that there is an isomorphism at all follows from Theorem 1.2. In order to get approximate unitary equivalence, we do the following.

Enumerate dense subsets of  $A, B$  as  $(a_\lambda)_{\lambda < \omega_1}$  and  $(b_\lambda)_{\lambda < \omega_1}$ . By transfinite induction we will show that there exists  $c_\lambda \in A, d_\lambda \in B$  and a unitary  $u_\lambda \in B$  for each  $\lambda < \omega_1$ , such that, for each  $\mu < \lambda$ ,

$$(1) \quad u_\lambda b_\mu u_\lambda^* = c_\mu \quad \text{and} \quad u_\lambda d_\lambda u_\lambda^* = a_\mu.$$

Note: at stage  $\lambda$ , we construct  $u_\lambda$  and, when applicable,  $c_{\lambda-1}$  and  $d_{\lambda-1}$ .

Let us describe stage  $\lambda$ , depending on whether  $\lambda$  is a successor or limit ordinal.

If  $\lambda = \lambda' + 1$ , we set  $S := \{c_\mu, a_\mu \mid \mu < \lambda\}$  (which is countable), and use the hypothesis to find a unitary  $v \in B \cap S'$  such that  $vu_{\lambda'} b_{\lambda'} u_{\lambda'}^* \in A$ . We set  $u_\lambda := vu_{\lambda'}$  and then set

$$c_{\lambda'} := u_\lambda b_{\lambda'} u_\lambda^* \in A, \quad d_{\lambda'} := u_\lambda^* a_{\lambda'} u_\lambda^*.$$

For  $\mu < \lambda$ , either  $\mu = \lambda'$  in which case clearly (1) holds, or else  $\mu < \lambda'$ , in which case (1) still holds since  $v$  commutes with  $S$ .

Now, suppose  $\lambda$  is a limit ordinal. Consider the set  $\Sigma$  of formulas in the variable  $u$ , consisting of  $\|u^*u - 1\|, \|uu^* - 1\|, \|ub_\mu u^* - c_\mu\|$  and  $\|ud_\mu u^* - a_\mu\|$ , for  $\mu < \lambda$ . These formulas are over a countable set of parameters, since  $\lambda < \omega_1$ , and the induction hypothesis shows that  $\Sigma$  is consistent. By countable saturation, it follows that there exists a unitary  $u_\lambda \in B$  realizing  $p$ ; this says that (1) holds.

This concludes the construction (existence proof) of  $c_\lambda, d_\lambda$  and  $u_\lambda$ .

We now define  $\Phi : A \rightarrow B$  by  $\Phi(a_\lambda) = d_\lambda$ . By construction  $\Phi$  is approximately unitarily equivalent to the inclusion  $A \rightarrow B$ , and it follows that it is an injective  $*$ -homomorphism. We need only show that it is surjective, and to do this, we show that  $\Phi(c_\lambda) = b_\lambda$ . Certainly, there exists  $\mu > \lambda$  such that  $a_\mu = c_\lambda$ , and therefore,

$$\Phi(c_\lambda) = d_\mu = u_{\mu+1}^* a_\mu u_{\mu+1} = u_{\mu+1}^* c_\lambda u_{\mu+1} = b_\lambda.$$

□

**Corollary 2.13.** *Suppose that  $D$  is a unital separable  $C^*$ -algebra with approximately inner half-flip,  $C$  is a countably saturated unital potentially  $D$ -absorbing  $C^*$ -algebra, and  $D$  is a unital subalgebra of  $C$ . Then*

- (1)  $C^*(D, D' \cap C) \otimes 1 \prec C \otimes D$  and the inclusion is approximately unitarily equivalent to an isomorphism between  $C^*(D, D' \cap C)$  and  $(D' \cap C) \otimes D$ ,
- (2)  $D' \cap C \prec C$ , and
- (3) if the continuum hypothesis holds and  $C$  has density character  $\aleph_1$  then this inclusion is approximately unitarily equivalent to an isomorphism.

In particular, if  $\mathcal{F}$  is a filter on  $\mathbb{N}$  such that  $\ell_\infty(A)/c_{\mathcal{F}}(A)$  is countably saturated and  $A$  is potentially  $D$ -absorbing then

$$D' \cap \ell_\infty(A)/c_{\mathcal{F}}(A) \prec A^{\mathcal{F}},$$

and if the continuum hypothesis holds then this embedding is approximately unitarily equivalent to an isomorphism.

*Proof.* We prove (2) and (3) first. For  $D' \cap C$  and  $C$ , we will use Theorem 2.12. Therefore, let  $S \subseteq D' \cap C$  be separable and let  $b \in C$ . Let  $\Psi : D \rightarrow S' \cap D' \cap C$  be an embedding.  $S' \cap C$  is  $D$ -saturated and countably quantifier-free saturated, so by Lemma 2.8, there exists a unitary  $u \in S' \cap C$  such that  $u^*Du = \Psi(D)$ . It follows that  $[ubu^*, D] = u[b, u^*Du]u^* = u[b, \Psi(D)]u^* = 0$ , i.e.,  $ubu^* \in D' \cap C$ . This verifies the hypotheses of Theorem 2.12 and so  $D' \cap C \prec C$  and since  $C$  is countably saturated, so is  $D' \cap C$ . If  $C$  has density character  $\aleph_1$  and the continuum hypothesis holds then inclusion  $D' \cap C \rightarrow C$  is approximately unitarily equivalent to an isomorphism.

For  $C^*(D' \cap C, D) \otimes 1$  and  $C \otimes D$ , note that since  $D$  has approximately inner half-flip, the embedding  $C^*(D' \cap C, D) \otimes 1_D \rightarrow C \otimes D$  is approximately unitarily equivalent, by unitaries in  $D \otimes D \subseteq C \otimes D$ , to an isomorphism  $C^*(D' \cap C, D) \rightarrow (D' \cap C) \otimes D$ . From the proof of (2),  $(D' \cap C) \otimes D \prec C \otimes D$ . Composing these, the inclusion  $C^*(D' \cap C, D) \otimes 1 \rightarrow C \otimes D$  is an elementary map.  $\square$

The previous result can be somewhat refined. If  $A$  is a C\*-algebra and  $B$  is a C\*-subalgebra, then we can consider the pair  $(A, B)$  as a model in an appropriate expansion of the language of C\*-algebras. To the standard language we add the unary predicate  $\text{dist}_B$ , the distance from  $a \in A$  to  $B$ . This function is 1-Lipshitz and therefore can be added to the language.

**Corollary 2.14.** *Assume  $D$  is a separable C\*-algebra with approximately inner half-flip,  $C$  is a C\*-algebra that is countably saturated, potentially  $D$ -absorbing, and that  $D \subseteq C$ . Then  $(C^*(D, D' \cap C), D' \cap C)$  can be elementarily embedded into  $(C \otimes D, C \otimes 1)$ .*

*Proof.* Looking at the proof of the previous corollary, we see that

$$((D' \cap C) \otimes D, (D' \cap C) \otimes 1) \prec (C \otimes D, C \otimes 1)$$

We also have

$$(C^*(D' \cap C, D), D' \cap C) \cong ((D' \cap C) \otimes D, (D' \cap C) \otimes 1)$$

and so the conclusion follows.  $\square$

**2.3. A model-theoretic characterization of strongly self-absorbing algebras.** The hyperfinite  $\text{II}_1$  factor is the only  $\text{II}_1$  factor with a separable predual all of whose embeddings into its ultrapower are conjugate ([22]). We give a similar characterization of strongly self-absorbing algebras.

**Theorem 2.15.** *A C\*-algebra  $D$  is strongly self-absorbing if and only if the following hold.*

- (1) *All unital \*-homomorphisms of  $D$  into its ultrapower  $D^{\mathcal{U}}$  are unitarily conjugate, and*
- (2)  *$D \equiv D \otimes D$ .*



*Proof.* Assume  $D$  is strongly self-absorbing. Then  $D \cong D \otimes D$  and therefore (2) holds, and (1) holds by Lemma 2.8.

Now assume (1) and (2). By (2),  $D \otimes D$  embeds elementarily into  $D^\mathcal{U}$  and by (1),  $D \otimes 1$  is unitarily conjugate to the diagonal copy of  $D$  embedded into  $D^\mathcal{U}$  so we may assume

$$D \prec D \otimes D \prec D^\mathcal{U}$$

where the first embedding is  $D \rightarrow D \otimes 1$ . Now  $1 \otimes D \subseteq D' \cap D^\mathcal{U}$  and by (1),  $D \otimes 1$  and  $1 \otimes D$  are unitarily conjugate so  $D$  has approximately inner half-flip. So by Theorem 2.1, we have  $D \cong D \otimes D$  say by an isomorphism  $\Phi$ . But again by (1),  $\Phi$  can be implemented by a unitary in  $D^\mathcal{U}$ . Since  $D \otimes D \prec D^\mathcal{U}$ ,  $\Phi$  is approximately unitarily equivalent to the map  $D \rightarrow D \otimes 1$  and so  $D$  is strongly self-absorbing.  $\square$

Unlike the case of  $\text{II}_1$  factors, the assumption that all unital  $*$ -homomorphisms of  $A$  into its ultrapower are unitarily conjugate alone does not imply that  $A$  is strongly self-absorbing or even that it is self-absorbing (i.e., that  $A \otimes A \cong A$ ). For example, every UHF algebra has this property so it suffices to take a UHF algebra that is not of infinite type. There are even  $C^*$ -algebras without approximately inner half-flip such that all of their  $*$ -homomorphisms into their ultrapowers are unitarily conjugate (one such algebra is  $\mathcal{O}_3$ ; see Example 3.8). If  $D$  has approximately inner half-flip then any two unital  $*$ -homomorphisms of  $D$  into  $D^\mathcal{U}$  are unitarily conjugate in  $D^\mathcal{U} \otimes D$  (by the proof of Lemma 2.8) but not necessarily in  $D^\mathcal{U}$  (Example 3.10).

**2.4. Classification.** Recall that a model  $D$  is *prime* if for every  $A \equiv D$  there is an elementary embedding of  $D$  into  $A$ . For separable models this is equivalent to being *atomic* (see [1]). In order to avoid confusion with terminology established in  $C^*$ -algebras (and since we are using the concept only for separable algebras) we shall refer to prime models as atomic models.

**Proposition 2.16.** *Every strongly self-absorbing algebra  $D$  is an atomic model of its theory.*

*Proof.* By the Downward Löwenheim–Skolem theorem, it will suffice to show that  $D$  elementarily embeds into every separable  $A \equiv D$ . Fix such an  $A$ . Then since  $A \equiv D$  we have  $A$  is  $D$ -absorbing and we can assume that  $A \prec D^\mathcal{U}$ . But since  $A \cong A \otimes D$ , we also have an embedding of  $D$  into  $A$ . Since all copies of  $D$  embedded in  $D^\mathcal{U}$  are conjugate to the diagonal embedding, any embedding of  $D$  into  $A$  is elementary.  $\square$

It is well-known that elementarily equivalent separable atomic models are isomorphic, but we shall show that in case of strongly self-absorbing algebras a stronger result is true.

A quantifier-free formula  $\psi(\bar{x})$  is called an  $\mathbb{R}^+$ -formula if for all  $C^*$ -algebras  $A$  and all  $\bar{a} \in A$ ,  $\psi^A(\bar{a}) \geq 0$ . A sentence  $\phi$  is *universal* if it is

of the form  $\sup_{\bar{x}} \psi(\bar{x})$  for some quantifier-free  $\mathbb{R}^+$ -formula  $\psi(\bar{x})$ . The *universal theory* of a C\*-algebra is

$$\text{Th}_{\forall}(A) = \{\phi : \phi^A = 0 \text{ and } \phi \text{ is universal}\}.$$

This terminology is adopted in order to match discrete first order logic: for a universal sentence  $\phi = \sup_{\bar{x}} \psi(\bar{x})$  and any C\*-algebra  $A$ ,  $\phi^A = 0$  if and only if for all  $\bar{a} \in A$ ,  $\psi^A(\bar{a}) = 0$ . If  $A$  is a subalgebra of  $B$  then clearly  $\text{Th}_{\forall}(A) \supseteq \text{Th}_{\forall}(B)$ . For separable  $A$  and countably saturated  $C$ , one has that  $A$  embeds into  $C$  if and only if  $\text{Th}_{\forall}(A) \supseteq \text{Th}_{\forall}(C)$  ([15]).

The following result was announced in [10].

**Theorem 2.17.** *Assume  $D$  and  $E$  are strongly self-absorbing.*

- (1)  $D$  is  $E$ -absorbing if and only if  $\text{Th}_{\forall}(D) \subseteq \text{Th}_{\forall}(E)$ .
- (2)  $D \cong E$  if and only if  $\text{Th}_{\forall}(D) = \text{Th}_{\forall}(E)$ .

*Proof.* (1) Only the converse implication requires a proof. Assume  $\text{Th}_{\forall}(D) \subseteq \text{Th}_{\forall}(E)$ . Since  $D \equiv D' \cap D^{\mathcal{U}}$ , we have  $\text{Th}_{\forall}(D) = \text{Th}_{\forall}(D' \cap D^{\mathcal{U}})$ . Since  $D' \cap D^{\mathcal{U}}$  is countably saturated,  $E$  embeds into  $D' \cap D^{\mathcal{U}}$  and therefore  $D \otimes E \cong D$  by Theorem 2.1.

(2) Again only the converse implication requires a proof. By using (1) twice we have that  $D \cong D \otimes E \cong E$ .  $\square$

We record a consequence on the classification problem for strongly self-absorbing algebras (see [17] for the definitions). While the isomorphism relation for separable C\*-algebras (and even separable AI algebras) is not classifiable by countable structures ([18]), our result shows that the isomorphism relation for strongly self-absorbing algebras is much simpler. Since the computation of a theory of a C\*-algebra is given by a Borel function ([17]) the following is an immediate consequence of Theorem 2.17.

**Corollary 2.18.** *The isomorphism relation of strongly self-absorbing algebras is smooth.*  $\square$

On the other hand, C\*-algebras with approximately inner half-flip behave much differently: Elementarily equivalent separable C\*-algebras with approximately inner half-flip are not necessarily isomorphic, the isomorphism relation of these C\*-algebras is not smooth, and they are not necessarily atomic models of their theories (Example 3.7).

### 3. LIMITING EXAMPLES

Examples promised earlier on are collected in this concluding section. Most interesting examples involve nontrivial properties of Kirchberg algebras reviewed in §3.1.

Examples 3.1 and 3.8 show two directions in which Theorem 2.15 cannot be improved.

**Example 3.1.** A separable unital nuclear  $C^*$ -algebra  $A$  such that  $A \cong A \otimes A$  and the images of all unital  $*$ -homomorphisms of  $A$  into  $A^{\mathcal{U}}$  are conjugate by an automorphism of  $A^{\mathcal{U}}$ , but  $A$  does not have approximately inner half-flip.

Take  $A$  to be  $C(\{0, 1\}^{\mathbb{N}})$ , the  $C^*$ -algebra of continuous functions on the Cantor space. Then  $A \otimes A \cong A$  since  $\{0, 1\}^{\mathbb{N}}$  is homeomorphic to its own square. Since the theory of  $A$  allows elimination of quantifiers ([8]) every embedding of  $A$  into a model of its theory is elementary and therefore the standard back-and-forth argument shows that any two such embeddings into a saturated model are conjugate.

If one considers  $A$ , a UHF algebra not of infinite type, then we have that every unital  $*$ -homomorphism of  $A$  into  $A^{\mathcal{U}}$  is conjugate by a unitary but  $A \not\cong A \otimes A$ .

The following example is well-known but is included for completeness.

**Example 3.2.** A free filter  $\mathcal{F}$  on  $\mathbb{N}$  such that  $\ell_{\infty}(A)/c_{\mathcal{F}}(A)$  is not countably quantifier-free saturated (and therefore fails Kirchberg's  $\varepsilon$ -test, [25, Lemma 3.1]) for any unital, simple  $C^*$ -algebra  $A$ .

Recall that  $\mathcal{Z}_0$  is the ideal of sets of asymptotic density zero,

$$\mathcal{Z}_0 := \{X \subseteq \mathbb{N} : \lim_n |X \cap \{0, \dots, n-1\}|/n = 0\}.$$

Define the upper density on  $\mathcal{P}(\mathbb{N})$  by

$$d(X) := \limsup_n |X \cap \{0, \dots, n-1\}|/n.$$

Then  $\mathcal{Z}_0$  is the ideal of sets of asymptotic density zero and the dual filter is equal to

$$\mathcal{F}_0 = \{X : d(\mathbb{N} \setminus X) = 0\}.$$

The following is an elaboration of the well-known fact that the quotient Boolean algebra  $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$  is not countably saturated.

Let  $X_n = \{j2^n : j \in \mathbb{N}\}$ . Then  $d(X_n) = 2^{-n}$  and  $X_n \supseteq X_{n+1}$ . If  $Y \subseteq \mathbb{N}$  is such that  $Y \setminus X \in \mathcal{Z}_0$  then  $d(Y) = 0$  and therefore  $\mathbb{N} \setminus Y \in \mathcal{F}_0$ . Let  $p_n$  be the projection in  $\ell_{\infty}(A)/c_{\mathcal{F}}(A)$  whose representing sequence satisfies  $p_n(j) = 1$  if  $j \in X_n$  and  $p_n(j) = 0$  if  $j \notin X_n$ . Then  $p_n$ , for  $n \in \mathbb{N}$ , is a strictly decreasing sequence of projections in  $\ell_{\infty}(A)/c_{\mathcal{F}}(A)$  and the type of a nonzero projection  $q$  such that  $q \leq p_n$  for all  $n$  is consistent, but not realized, in  $\ell_{\infty}(A)/c_{\mathcal{F}}(A)$ .

The argument in Example 3.2 shows that  $\ell_{\infty}(A)/c_{\mathcal{F}}(A)$  is not countably degree-1 saturated (see [11]) and even not SAW\* ([30]). No example of a  $C^*$ -algebra which is countably quantifier free saturated but not countably saturated is known (see [8]). The Calkin algebra is an example of a  $C^*$ -algebra that is countably degree-1 saturated but not quantifier-free saturated ([11, §4]).

**3.1. Kirchberg algebras.** We review some facts on Kirchberg algebras. More details, including an exposition of Kirchberg and Phillips' classification result, can be found in e.g., [33]. A separable, simple, purely infinite and nuclear C\*-algebra is called a *Kirchberg algebra*. Phillips and Kirchberg have classified all Kirchberg algebras up to  $KK$ -theory, and unital Kirchberg algebras in the so-called UCT class are classified by their  $K$ -groups together with the position of the unit in  $K_0$ . To each pair  $(G_0, G_1)$  of countable abelian groups there is a Kirchberg algebra  $A$  in the UCT class with  $K_0(A) \cong G_0$  and  $K_1(A) \cong G_1$ .

The classification theorem contains the following fact (see [31, Theorem 4.4.1] or [33, Theorem 8.2.1]) that we shall use. The last statement of the theorem follows using [33, Remark 2.4.8].

**Theorem 3.3** (Kirchberg, Phillips). *If  $A$  and  $B$  are unital Kirchberg algebras and if  $\varphi, \psi: A \rightarrow B$  are unital \*-homomorphisms, then  $\varphi$  is asymptotically unitarily equivalent to  $\psi$  if and only if  $KK(\varphi) = KK(\psi)$  in  $KK(A, B)$ . If  $A$  and  $B$  are in the UCT class and with finitely generated  $K$ -groups, then this, in turn, is equivalent to  $\varphi$  and  $\psi$  being approximately unitarily equivalent.*

A Kirchberg algebra  $A$  is said to be in *standard form* if  $A$  is unital and  $[1_A] = 0$  in  $K_0(A)$ . Every Kirchberg algebra is Morita (or stably) equivalent to a Kirchberg algebra in standard form, and the standard form is unique up to isomorphism. Any C\*-algebra which is Morita equivalent to a Kirchberg algebra  $A$  and which is in standard form is isomorphic to  $p(A \otimes K)p$  for some projection  $p \in A \otimes K$  with  $[p] = 0$  in  $K_0(A)$  (and such a projection  $p$  always exists, in fact in  $A$ ). The uniqueness of the standard form is deduced from the elementary fact that if  $p$  and  $q$  are non-zero projections in  $A \otimes K$  with  $[p] = [q]$  in  $K_0(A)$ , then  $p \sim q$ .

It is a well-known fact that  $A$  is in standard form if and only if the Cuntz algebra  $\mathcal{O}_2$  embeds unitaly into  $A$ . Indeed, "if" follows from the fact that  $K_0(\mathcal{O}_2) = 0$ . Conversely, if  $A$  is a Kirchberg algebra in standard form, then  $2[1_A] = [1_A]$ , which implies that  $1_A = p + q$  for some non-zero projections  $p, q \in A$  with  $p \sim q \sim 1_A$  (again using that  $[p] = [q]$  implies  $p \sim q$ ). Let  $s_1, s_2 \in A$  be such that  $s_1^*s_1 = s_2^*s_2 = 1_A$ ,  $s_1s_1^* = p$  and  $s_2s_2^* = q$ . Then  $C^*(s_1, s_2)$  is a unital sub-C\*-algebra of  $A$  which is isomorphic to  $\mathcal{O}_2$ .

**Lemma 3.4.** *Every Kirchberg algebra  $D$  in standard form has approximately inner half-flip and satisfies  $D^{\otimes \infty} \cong \mathcal{O}_2$ .*

*Proof.* We must show that the \*-homomorphisms  $\alpha(d) = d \otimes 1_D$  and  $\beta(d) = 1_D \otimes d$ ,  $d \in D$ , from  $D$  to  $D \otimes D$  are approximately unitarily equivalent. Since  $D$  and  $D \otimes D$  are Kirchberg algebras we can use the Kirchberg–Phillips' classification theorem (Theorem 3.3) whereby it suffices to show that  $KK(\alpha) = KK(\beta)$  in  $KK(D, D \otimes D)$ . Since  $D$  is in standard form there is a unital embedding  $\mathcal{O}_2 \rightarrow D$ . We can therefore factor  $\alpha$  and  $\beta$

through  $\mathcal{O}_2$  as follows

$$D \rightarrow D \otimes \mathcal{O}_2 \rightarrow D \otimes D, \quad D \rightarrow \mathcal{O}_2 \otimes D \rightarrow D \otimes D,$$

where we recall that  $D \otimes \mathcal{O}_2 \cong \mathcal{O}_2 \otimes D \cong \mathcal{O}_2$  by [24, Theorem 3.8]. As  $KK(\mathcal{O}_2, \mathcal{O}_2) = 0$ , this implies that  $KK(\alpha) = KK(\beta) = 0$ .

As for the second claim, arguing as above, we can write  $D^{\otimes \infty}$  as the inductive limit of the sequence:

$$D \rightarrow D \otimes \mathcal{O}_2 \rightarrow D \otimes D \rightarrow D \otimes D \otimes \mathcal{O}_2 \rightarrow D \otimes D \otimes D \rightarrow \dots$$

Every other C\*-algebra in this sequence is isomorphic to  $\mathcal{O}_2$  (by [24, Theorem 3.8]), whence  $D^{\otimes \infty}$  is an inductive limit of a sequence of copies of  $\mathcal{O}_2$ ; and the inductive limit of such a sequence is isomorphic to  $\mathcal{O}_2$  (by the Kirchberg-Phillips' classification or by [32]).  $\square$

It is known that  $\mathcal{O}_2$  and  $\mathcal{O}_\infty$  are strongly self-absorbing (see e.g., [33]), so in particular they have approximately inner flip.

For the reader's convenience we reproduce the following well-known result due to Cuntz ([7]) and sketch its proof.

**Proposition 3.5.** *The Cuntz algebras  $\mathcal{O}_n$  do not have approximately inner half-flip when  $2 < n < \infty$ .*

*Proof.* Let  $\alpha, \beta: \mathcal{O}_n \rightarrow \mathcal{O}_n \otimes \mathcal{O}_n$  be the two canonical endomorphisms given by  $\alpha(d) = d \otimes 1$  and  $\beta(d) = 1 \otimes d$ ,  $d \in \mathcal{O}_n$ . The interrelations between  $\alpha$  and  $\beta$  are encoded in the unitary

$$u = \sum_{j=1}^n \alpha(s_j)\beta(s_j)^* = \sum_{j=1}^n s_j \otimes s_j^* \in \mathcal{O}_n \otimes \mathcal{O}_n,$$

cf. [32], where  $s_1, s_2, \dots, s_n$  are the canonical generators of  $\mathcal{O}_n$ . It is well-known that  $\alpha$  and  $\beta$  are not unitarily equivalent if  $[u] \notin (n-1)K_1(\mathcal{O}_n \otimes \mathcal{O}_n)$  (see [32, Theorem 3.6]). We proceed to prove this fact.

With  $B$  denoting the UHF algebra  $M_{n^\infty}$  we can identify  $\mathcal{O}_n$  with the crossed product  $B \rtimes_\rho \mathbb{N}$  where  $\rho(a) = s_1 a s_1^*$  (see the proof of [33, Theorem 4.2.2]). Therefore  $\mathcal{O}_n \otimes \mathcal{O}_n$  is identified with  $(\mathcal{O}_n \otimes B) \rtimes_{\rho'} \mathbb{N}$  where  $\rho'(a \otimes b) := a \otimes \rho(b)$ . By stability and continuity of  $K$ -theory it is easy to see that  $K_j(\mathcal{O}_n \otimes B) \cong K_j(\mathcal{O}_n) \otimes K_0(B)$  for  $j = 0, 1$ , and so  $K_0(\mathcal{O}_n \otimes B) \cong \mathbb{Z}/(n-1)\mathbb{Z}$  and  $K_1(\mathcal{O}_n \otimes B) = 0$ . Therefore the Pimsner-Voiculescu exact sequence ([3, Theorem 10.2.2]) for this crossed product becomes:

$$\begin{array}{ccccc} \mathbb{Z}/(n-1)\mathbb{Z} & \xrightarrow{0} & \mathbb{Z}/(n-1)\mathbb{Z} & \xrightarrow{K_0(\iota)} & K_0(\mathcal{O}_n \otimes \mathcal{O}_n) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(\mathcal{O}_n \otimes \mathcal{O}_n) & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

(with  $\iota: \mathcal{O}_n \otimes B \rightarrow \mathcal{O}_n \otimes \mathcal{O}_n$  denoting the inclusion map and  $\delta_1$  denoting the index map). By the exactness of this sequence at the bottom left corner we see that  $K_1(\mathcal{O}_n \otimes \mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ . Since  $K_0(\mathcal{O}_n \otimes B)$  is generated by the

class of the unit (as in the case of  $K_0(\mathcal{O}_n)$ ), a (partial) unitary  $w \in \mathcal{O}_n \otimes \mathcal{O}_n$  is a generator for  $K_1(\mathcal{O}_n \otimes \mathcal{O}_n)$  if and only if  $\delta_1([w]) = [1]$ .

With  $u$  as above,  $v := u^*(1 \otimes s_1^*) = \sum_{j=1}^n s_j^* \otimes s_j s_1^*$  is a co-isometry in  $\mathcal{O}_n \otimes B$ , and  $u^* = v(1 \otimes s_1)$ . Thus, by the definition of the index map  $\delta_1$  in the Pimsner–Voiculescu six term exact sequence above, one can conclude that  $\delta_1([u^*]) = [1]$ , whence  $[u]$  is a generator for  $K_1(\mathcal{O}_n \otimes \mathcal{O}_n)$ . In particular,  $[u] \neq 0$  and  $\mathcal{O}_n$  does not have approximately inner half-flip  $\square$

**3.2. Limiting examples from Kirchberg algebras.** This example contains limiting examples, whose interesting properties are derived from the Kirchberg–Phillips classification of Kirchberg algebras (Theorem 3.3). After this article was submitted, the last-named author undertook a more thorough investigation of the relationship between approximately inner flip and  $K$ -theory, describing exactly which UCT-satisfying Kirchberg algebras (and which stably finite, classifiable C\*-algebras) have approximately inner flip [34]; that paper is largely complementary to the present examples.

It is well-known that C\*-algebras with approximately inner flip need not be strongly self-absorbing. For example, matrix algebras and UHF algebras not of infinite type have this property.

**Example 3.6.** For every  $m \geq 2$  there exists a separable unital C\*-algebra  $A$  such that  $A$  does not have approximately inner half-flip and  $M_m(A)$  has approximately inner half-flip but not approximately inner flip.

Proposition 3.5 implies that  $\mathcal{O}_n$  does not have approximately inner half-flip and therefore  $M_{n+1}(\mathcal{O}_n)$  does not have approximately inner flip by [9, Corollary 2.6]. Because it is in standard form,  $M_{n+1}(\mathcal{O}_n)$  has approximately inner half-flip by Lemma 3.4.

**Example 3.7.** Not every separable unital C\*-algebra with approximately inner half-flip is an atomic model of its theory. Moreover there are elementarily equivalent separable C\*-algebras with approximately inner half-flip which are not isomorphic, and the isomorphism relation for separable C\*-algebras with approximately inner half-flip is not smooth. (The examples here are, in addition, simple.)

Lemma 3.4 implies that every Kirchberg algebra in standard form has approximately inner half-flip and therefore provides a large supply of Kirchberg algebras with approximately inner half-flip. By [19, Theorem 5.7.1], the map that associates  $K_0$  and  $K_1$  to a separable C\*-algebra is a Borel map from the Borel space of separable C\*-algebras to the Borel space of pairs of countable abelian groups (in this context, the order is trivial and we ignore the  $K_0$ -class of the identity). By the Kirchberg–Phillips classification and corresponding range of invariant theorem (see [33]) this map, restricted to the category of Kirchberg algebras satisfying the UCT, is equivalence of categories. Since the isomorphism of countable abelian groups is not a smooth equivalence relation, there are elementarily equivalent but non-isomorphic

UCT Kirchberg algebras in the standard form. Since elementarily equivalent separable atomic models are isomorphic, some of these C\*-algebras are not atomic.

This proof shows that there are non-atomic C\*-algebras with approximately inner half-flip with trivial  $K_1$  and torsion-free  $K_0$  of rank 2, since the isomorphism of torsion-free rank 2 groups is not smooth.

The argument in Example 3.7 is nonconstructive and very similar to the proof that there are elementarily equivalent but nonisomorphic separable AF algebras ([5, Theorem 3 (1)]). In both cases we don't have an explicit natural example of elementarily equivalent but nonisomorphic C\*-algebras. We also do not know whether elementarily equivalent separable C\*-algebras with approximately inner flip are necessarily isomorphic.

**Example 3.8.** A separable unital simple nuclear C\*-algebra  $A$  such that all unital \*-homomorphisms of  $A$  into  $A^{\mathcal{U}}$  are unitarily conjugate but  $A$  does not have approximately inner half-flip. (This differs from Example 3.1 in that  $A$  is simple,  $A \not\cong A \otimes A$ , and that \*-homomorphisms from the C\*-algebra in Example 3.1 to its ultraproduct are not all unitarily conjugate.)

There are many examples; we take  $\mathcal{O}_3$ . By the Universal Coefficient Theorem ([3, Theorem 23.1.1]), and since  $K_1(\mathcal{O}_3) = 0$ , we have

$$KK(\mathcal{O}_3, \mathcal{O}_3) \cong \text{Hom}(K_0(\mathcal{O}_3), K_0(\mathcal{O}_3)).$$

As  $K_0(\mathcal{O}_3) \cong \mathbb{Z}/2\mathbb{Z}$  and  $[1]$  is a generator for  $K_0(\mathcal{O}_3)$  it follows that  $KK(\varphi) = \mathbf{1}$  for all unital endomorphisms  $\varphi$  on  $\mathcal{O}_3$ . Hence, by the Kirchberg–Phillips classification theorem (Theorem 3.3) any two unital endomorphisms of  $\mathcal{O}_3$  are approximately unitarily equivalent.

Since  $\mathcal{O}_3$  is semiprojective ([2]) every \*-homomorphism  $\Phi: \mathcal{O}_3 \rightarrow \mathcal{O}_3^{\mathcal{U}}$  lifts to a \*-homomorphism  $\tilde{\Phi}: \mathcal{O}_3 \rightarrow \ell_\infty(\mathcal{O}_3)$ . Fix \*-homomorphisms  $\Phi_1$  and  $\Phi_2$  of  $\mathcal{O}_3$  into  $\mathcal{O}_3^{\mathcal{U}}$ . By the above there is a sequence of unitaries  $u_n$  in  $\ell_\infty(\mathcal{O}_3)$  whose images under the quotient map witness that  $\Phi_1$  is approximately unitarily equivalent to  $\Phi_2$ . By the countable saturation we can find a single unitary  $u \in \mathcal{O}_3^{\mathcal{U}}$  such that  $\Phi_2 = \text{Ad } u \circ \Phi_1$ , as required.

Our last example uses the following version of stability of  $KK$ :

**Lemma 3.9.** *Let  $A$  and  $B$  be C\*-algebras and let  $n \geq 1$  be an integer. It follows that there is an isomorphism*

$$\rho: KK(A, B) \rightarrow KK(A \otimes M_n, B \otimes M_n)$$

*such that  $\rho(KK(\varphi)) = KK(\varphi \otimes \text{id}_n)$  for all \*-homomorphisms  $\varphi: A \rightarrow B$ .*

*Proof.* For each C\*-algebra  $D$ , let  $\iota_{D,n}: D \rightarrow D \otimes M_n$  denote the canonical inclusion  $\iota_{D,n}(d) = d \otimes e_{11}$ . Then  $KK(\iota_{D,n})$  defines an invertible element of  $KK(D, D \otimes M_n)$  by stability of  $KK$ -theory. Hence the map

$$\rho: KK(A, B) \rightarrow KK(A \otimes M_n, B \otimes M_n), \quad x \mapsto KK(\iota_{A,n})^{-1} \cdot x \cdot KK(\iota_{B,n}),$$

where “ $\cdot$ ” denotes the Kasparov product, is an isomorphism. Inspection shows that  $\iota_{A,n} \circ \varphi = (\varphi \otimes \text{id}_n) \circ \iota_{B,n}$  for each  $*$ -homomorphism  $\varphi: A \rightarrow B$ , which in particular implies that

$$KK(\iota_{A,n}) \cdot KK(\varphi) = KK(\varphi \otimes \text{id}_n) \cdot KK(\iota_{B,n}).$$

This completes the proof.  $\square$

**Example 3.10.** A separable C\*-algebra  $D$  with approximately inner flip such that  $D \otimes D \cong D$  but there exists an automorphism  $\phi$  of  $D$  that is not approximately inner. In particular,

- (1) not all unital  $*$ -homomorphisms of  $D$  into  $D^{\mathcal{U}}$  are unitarily conjugate,
- (2)  $D^{\mathcal{U}}$  is not potentially  $D$ -absorbing,
- (3)  $D$  does not embed into  $D' \cap D^{\mathcal{U}}$  (and therefore  $D$  does not satisfy  $\mathbb{T}_D$ ),
- (4)  $D$  is not s.s.a, and
- (5) although  $\phi$  and  $\text{id}_D$  are not approximately unitarily equivalent, their compositions with the first-factor inclusion  $D \rightarrow D \otimes D$  are approximately unitarily equivalent.

Let  $D$  be the standard form of  $\mathcal{O}_\infty$ , that is  $D = p\mathcal{O}_\infty p$  where  $p$  is a projection in  $\mathcal{O}_\infty$  with  $[p] = 0$  in  $K_0(\mathcal{O}_\infty)$ . Then  $D$  has approximately inner half-flip by Lemma 3.4. One could show, using stability of  $KK$  as in Lemma 3.9, that this C\*-algebra even has approximately inner flip. As  $D$  is stably isomorphic to  $\mathcal{O}_\infty$  we see that  $D \otimes D$  is stably isomorphic to  $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$  which again is isomorphic to  $\mathcal{O}_\infty$ . Moreover,  $D \otimes D$  is in standard form (because  $D$  and hence also  $D \otimes D$  admit unital embeddings of  $\mathcal{O}_2$ ). This entails that  $D \otimes D \cong D$ .

There is an automorphism of  $D$  which reverses  $K_0$ . It cannot be approximately inner, because approximately inner maps agree with the identity on  $K$ -theory.

Clause (2) now follows from Lemma 2.8 and then (3) follows from Proposition 2.7. (4) follows from Theorem 2.1.

To see (5), note that  $[\phi(p) \otimes 1_D] = 0 = [d \otimes 1_D]$  for all  $d \in D$ . Hence, these two maps from  $D$  to  $D \otimes D$  agree on  $K$ -theory. By the Universal Coefficient Theorem and since  $K_1(\mathcal{O}_\infty) = 0$ ,

$$KK(\mathcal{O}_\infty, \mathcal{O}_\infty) \cong \text{Hom}(K_0(\mathcal{O}_\infty), K_0(\mathcal{O}_\infty)).$$

Hence, by the Kirchberg–Phillips classification theorem (Theorem 3.3), these two maps are approximately unitarily equivalent.

## REFERENCES

- [1] I. Ben Yaacov, A. Berenstein, C.W. Henson, and A. Usvyatsov. Model theory for metric structures. In Z. Chatzidakis et al., editors, *Model Theory with Applications to Algebra and Analysis, Vol. II*, number 350 in London Math. Soc. Lecture Notes Series, pages 315–427. Cambridge University Press, 2008.
- [2] B. Blackadar. Shape theory for C\*-algebras. *Math. Scand.*, 56(2):249–275, 1985.



- [3] B. Blackadar. *K-theory for operator algebras*, volume 5. Cambridge University Press, 1998.
- [4] N.P. Brown. Topological dynamical systems associated to  $\text{II}_1$  factors. *Adv. Math.*, 227(4):1665–1699, 2011. With an appendix by Narutaka Ozawa.
- [5] K. Carlson, E. Cheung, I. Farah, A. Gerhardt-Bourke, B. Hart, L. Mezuman, N. Sequeira, and A. Sherman. Omitting types and AF algebras. *Arch. Math. Logic*, 53:157–169, 2014.
- [6] A. Connes. Classification of injective factors. Cases  $\text{II}_1$ ,  $\text{II}_\infty$ ,  $\text{III}_\lambda$ ,  $\lambda \neq 1$ . *Ann. of Math. (2)*, 104:73–115, 1976.
- [7] J. Cuntz. The internal structure of simple  $C^*$ -algebras. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 85–115. Amer. Math. Soc., Providence, R.I., 1982.
- [8] C. Eagle and A. Vignati. Saturation and elementary equivalence of  $C^*$ -algebras. *J. Funct. Anal.*, 269(8):2631–2664, 2015.
- [9] E. G. Effros and J. Rosenberg.  $C^*$ -algebras with approximately inner flip. *Pacific J. Math*, 77(2):417–443, 1978.
- [10] I. Farah. Logic and operator algebras. To appear in Proceedings of the Seoul ICM; arXiv:1404.4978, 2014.
- [11] I. Farah and B. Hart. Countable saturation of corona algebras. *C.R. Math. Rep. Acad. Sci. Canada*, 35:35–56, 2013.
- [12] I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, and W. Winter. Model theory of nuclear  $C^*$ -algebras. arXiv:1602.08072, 2016.
- [13] I. Farah, B. Hart, and D. Sherman. Model theory of operator algebras I: Stability. *Bull. London Math. Soc.*, 45:825–838, 2013.
- [14] I. Farah, B. Hart, and D. Sherman. Model theory of operator algebras II: Model theory. *Israel J. Math.*, 201:477–505, 2014.
- [15] I. Farah, B. Hart, and D. Sherman. Model theory of operator algebras III: Elementary equivalence and  $\text{II}_1$  factors. *Bull. London Math. Soc.*, 46:1–20, 2014.
- [16] I. Farah and S. Shelah. Rigidity of continuous quotients. *J. Math. Inst. Jussieu*, 15:1–28, 2016.
- [17] I. Farah, A.S. Toms, and A. Törnquist. The descriptive set theory of  $C^*$ -algebra invariants. *Int. Math. Res. Notices*, 22:5196–5226, 2013. Appendix with C. Eckhardt.
- [18] I. Farah, A.S. Toms, and A. Törnquist. Turbulence, orbit equivalence, and the classification of nuclear  $C^*$ -algebras. *J. Reine Angew. Math.*, 688:101–146, 2014.
- [19] E. Gardella and M. Lupini. Conjugacy and cocycle conjugacy of automorphisms of  $\mathcal{O}_2$  are not Borel. Accepted for publication in *Münster J. Math.*; arXiv:1404.3617, 2014.
- [20] S. Ghasemi. Reduced products of metric structures: a metric Feferman–Vaught theorem. arXiv:1411.0794, 2014.
- [21] S. Ghasemi.  $\text{SAW}^*$  algebras are essentially non-factorizable. *Glasg. Math. J.*, 57(1):1–5, 2015.
- [22] K. Jung. Amenability, tubularity, and embeddings into  $\mathbb{R}^\omega$ . *Math. Ann.*, 338(1):241–248, 2007.
- [23] E. Kirchberg. Central sequences in  $C^*$ -algebras and strongly purely infinite algebras. In *Operator Algebras: The Abel Symposium 2004*, volume 1 of *Abel Symp.*, pages 175–231. Springer, Berlin, 2006.
- [24] E. Kirchberg and N. C. Phillips. Embedding of exact  $C^*$ -algebras in the Cuntz algebra  $\mathcal{O}_2$ . *J. reine angew. Math.*, 525:17–53, 2000.
- [25] E. Kirchberg and M. Rørdam. Central sequence  $C^*$ -algebras and tensorial absorption of the Jiang–Su algebra. *J. Reine Angew. Math.*, 695:175–214, 2014.
- [26] E. Kirchberg and M. Rørdam. When central sequence  $C^*$ -algebras have characters. *Int. J. Math.*, 26(7):1550049 (32 pgs.), 2015.

- [27] H. Lin. Approximate unitary equivalence in simple C\*-algebras of tracial rank one. *Trans. Amer. Math. Soc.*, 364(4):2021–2086, 2012.
- [28] H. Matui and Y. Sato. Strict comparison and  $\mathcal{Z}$ -absorption of nuclear C\*-algebras. *Acta Math.*, 209(1):179–196, 2012.
- [29] D. McDuff. Central sequences and the hyperfinite factor. *Proc. London Math. Soc.*, 21:443–461, 1970.
- [30] G. K. Pedersen. The corona construction. In *Operator Theory: Proceedings of the 1988 GPOTS-Wabash Conference (Indianapolis, IN, 1988)*, volume 225 of *Pitman Res. Notes Math. Ser.*, pages 49–92. Longman Sci. Tech., Harlow, 1990.
- [31] N. C. Phillips. A classification theorem for nuclear purely infinite simple C\*-algebras. *Doc. Math.*, 5:49–114, 2000.
- [32] M. Rørdam. Classification of inductive limits of Cuntz algebras. *J. Reine Angew. Math.*, 440:175–200, 1993.
- [33] M. Rørdam. *Classification of nuclear C\*-algebras*, volume 126 of *Encyclopaedia of Math. Sciences*. Springer-Verlag, Berlin, 2002.
- [34] A. Tikuisis.  $K$ -theoretic characterization of C\*-algebras with approximately inner flip. *Int. Math. Res. Not. IMRN*, 2015. Art. ID rnv 334, 25 pages.
- [35] A. S. Toms and W. Winter. Strongly self-absorbing C\*-algebras. *Trans. Amer. Math. Soc.*, 359(8):3999–4029, 2007.

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