An asymmetric model of spatial competition.*

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Abstract

This paper explores a generalized spatial voting model in which parties are not supposed to be identical before the game. This new approach to the political market leads to substantial changes in parties'ss strategies. Our model provides new explanations of why parties may choose non median policies, i.e. other than that preferred by the median voter. It also provides explanations on why elections may not lead to close races.

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1 Introduction.

According to Downs (1957), political parties are mainly interested in winning per se and differ in their electoral platforms. The partisan politics approach (Wittman 1977, 1990) accept the idea that parties may have other goals but still assume that voters decide who to vote for according to the parties platforms. However, the retrospective voting approach[1] emphasizes the role played by other variables, such as the action held while on power, in the determination of the voters choice. This paper is an attempt to associate the retrospective approach and the work on party's strategic behavior.. Our model thus captures the fact that factors outside the parties control may affect the number of vote received by parties or candidates: one candidate could be more charismatic than the other, the incumbent may be viewed as more efficient than his/her challenger while in power, etc. All this information is supposed to be summarized in a single parameter, representing the quality of a party. Our model is asymmetric in the sense that we study competition between two parties that may have a different quality parameter.

The paper is organized as follows : in the next section we briefly recall existing results and show how our approach is situated in the stream of research on spatial models of voting. Section 3 presents the model and main assumptions and section 4 presents the main results.

2 Convergence in spatial models.

Some models aim to explain why parties may choose non median platforms (see Roemer 1994 for a survey). Alesina and Rosenthal (1995) show that two main criteria are needed in order to produce divergence at equilibrium. The first criterion is partisan politics as proposed by Wittman (1977, 1983): parties are not interested in winning *per se*, they defend a partisan view of the world. The second criterion is the existence of uncertainty in the behavior of the electorate: parties do not know with certainty where the median policy is located. By introducing an extra parameter that parties do not control, we obtain divergence at equilibrium *without* introducing any uncertainty. Furthermore, taking this additional parameter into account (one may consider this parameter as an indicator of the quality of the party) is more realistic since parties are not always engaged in close races.

[1]Main references are listed in Mueller (1995).

3 The model.

The parties. Our first assumption concerns the goal of parties. A usual way to model partisan politics (Roemer 1994) is to suppose that parties seek to minimize the distance between their preferred platform and the winning one. We use here a slightly different version called "light partisanship": parties are supposed to be mainly interested in winning the election, as in Downs model, but once they are assured of winning the election, they try to promote a program as close as possible to their preferred policy. This two-step definition rules out unsatisfactory equilibria in which a party may choose to lose the election in order to promote a policy closer to its ideal point[2]. An example of such an equilibria is presented in appendix.

Formally, a party, or candidate j, is represented by a triple $(\theta_j, p_j, x_j) \in \mathbb{R} \times X \times X$. θ_j represents the exogenous parameter, say the quality of candidate j, p_j its preferred policy and x_j denotes the platform it announces. The set of available platforms, X, is supposed to be a compact subset of $\mathbb{R}^n$. The proportion of voters who prefer party j to its opponent is given by the function $S_j(x_1, \theta_1, x_2, \theta_2)$. Thus, the light partisanship assumption is formally given by:

- **Assumption 1:**

$$v_j(x_1, x_2) = \begin{cases} S_j(x_1, \theta_1, x_2, \theta_2) & \text{if } S_j(x_1, \theta_1, x_2, \theta_2) \leq 1/2 \\ -\|x_j - p_j\| + C & \text{if } S_j(x_1, \theta_1, x_2, \theta_2) > 1/2 \end{cases}$$

C is a constant chosen in order to have parties preferring to win the election rather than losing it. That is to say, C is chosen so that $-\|x_j - p_j\| + C \geq 1/2$ for every $x_j \in X$. Note that we have no guaranty that v_j is a continuous function.

The electorate. A voter i has a utility function of the form $u(\theta_j, \|x_j - a_i\|)$ where $a_i \in X$ designates his bliss point. Voters' bliss points are distributed according to some probability distribution $f : X \to [0, 1]$ continuous with

[2]In the unidimensional setting, those equilibria are usually ruled out because of the assumption made on p_j. They are supposed to be on each side of the median. As we intend to address the more general multidimensional case, there are no straightforward assumptions on the p_j that eliminate such equilibria.

respect to Lebesgue measure (thus, $\int_X df = 1$). Other things being equal, a voter prefers a party that announces a platform closer to his bliss point and a higher quality party. At this stage, we have to make explicit the nature of the voters trade-off between quality and platforms. Following Mussa and Rosen (1978) and more generally the work on quality in industrial economics, we choose a multiplicatively separable form :

- **Assumption 2:** $u(\theta_j, \|x_j - a_i\|) = \theta_j \|x_j - a_i\|$, $\theta < 0$.

As usual in spatial models of voting, voters differs only in terms of their bliss points a_i. All of them arbitrate between quality and platforms in the same way. Thus, a voter will be identify only by its bliss point.

We define the support the parties as the sets :

$$\Omega_1 = \{a_i \in X \,|\theta_1 \|x_1 - a_i\| \geq \theta_2 \|x_2 - a_i\| \}$$
$$\text{and } \Omega_2 = \{a_i \in X \,|\theta_2 \|x_2 - a_i\| \geq \theta_1 \|x_1 - a_i\| \}$$

Thus, $S_j(x_1, \theta_1, x_2, \theta_2) = \int_{\Omega_j} df$.

The game of asymmetric competition. We define $k = \frac{\theta_2}{\theta_1}$ and assume, without loss of generality, that $\theta_2 < \theta_1 < 0$. Then party 1 gets a clear advantage. The game of asymmetric electoral competition is now defined and its main elements are summarized by the triple $(\mathbf{p}_1, \mathbf{k}, \mathbf{f})$.

4 Main results.

Before stating our main theorem, we prove a series of lemmas in order to explore the properties of the sets Ω_j. The basic idea is that in order to compute the score of a party, we need to explore the properties of the sets Ω_j. The reader will get much of the intuition of these results by referring to Figure I.

Lemma 1 ***The set*** $\Omega_2(x_1, \theta_1, x_2, \theta_2)$ ***is a hyperball centered at*** $c(x_1, \theta_1, x_2, \theta_2) = \frac{1}{1-k^2}x_1 - \frac{k^2}{1-k^2}x_2$ ***of radius*** $r(x_1, \theta_1, x_2, \theta_2) = \frac{k}{|1-k^2|} \|x_1 - x_2\|$.

- Proof.

Consider the set $I(x_1,\theta_1,x_2,\theta_2) = \{a_i \in X/\theta_1 \|x_1 - a_i\| = \theta_2 \|x_2 - a_i\|\}$ representing the boundary of the sets Ω_j i.e. the set of voters indifferent between the two parties.

This leads to the following equations :

$$\begin{array}{c}\theta_1 \|x_1 - a\| = \theta_2 \|x_2 - a\| \\ \Leftrightarrow \|x_1 - a\|^2 = k^2 \|x_2 - a\|^2\end{array}$$

Recalling the property of the euclidean inner product,
$\langle x_1 - a, x_1 - a\rangle = \|x_1\|^2 - 2\langle x_1, a\rangle + \|a\|^2$, we obtain:

$$\begin{array}{rcl}\|x_1\|^2 - 2\langle x_1, a\rangle + \|a\|^2 &=& k^2(\|x_2\|^2 - 2\langle x_2, a\rangle + \|a\|^2) \\ (1-k^2)\|a\|^2 - 2\langle a, x_1 - k^2 x_2\rangle &=& k^2\|x_2\|^2 - \|x_1\|^2 \\ \|a\|^2 - 2\left\langle a, \dfrac{x_1 - k^2 x_2}{1-k^2}\right\rangle &=& \dfrac{k^2\|x_2\|^2 - \|x_1\|^2}{1-k^2} \\ \left\| a - (\dfrac{x_1 - k^2 x_2}{1-k^2})\right\|^2 &=& \dfrac{k^2\|x_2\|^2 - \|x_1\|^2}{1-k^2} + \left\|(\dfrac{x_1 - k^2 x_2}{1-k^2})\right\|^2,\end{array}$$

Thus, $I(x_1,\theta_1,x_2,\theta_2)$ is the boundary of a hyperball of center $c(x_1,\theta_1,x_2,\theta_2) = \frac{1}{1-k^2}x_1 + \frac{-k^2}{1-k^2}x_2$ and radius $r^2 = \frac{k^2\|x_2\|^2 - \|x_1\|^2}{1-k^2} + \left\|(\frac{x_1-k^2x_2}{1-k^2})\right\|^2$. It is easy to check that $\Omega_2(x_1,\theta_1,x_2,\theta_2)$ consist of the whole hyperball.

After simplification[3]:

$$c = \beta x_1 + (1-\beta)x_2 \text{ where } \beta = \tfrac{1}{1-k^2} < 0$$

$$r = \tfrac{k}{k^2-1}\|x_1 - x_2\| \qquad \blacklozenge$$

Voters supporting the favored party have their bliss point included in a hyperball[4]. Our next lemma shows how the hyperball $\Omega_2(x_1,\theta_1,x_2,\theta_2)$ can be constructed.

Lemma 2 ***Suppose that*** x_2 ***is on a line*** Δ ***containing*** x_1. *The sets* $\Omega_2(x_1,\theta_1,x_2,\theta_2)$ *are then hyperballs tangent to the cone of* ***vertex*** x_1 ***and angle*** 2α ***such that*** $\sin\alpha = \frac{k}{|1-k^2|}$.

[3] The details of the calculations are given in the Appendix.

[4] Voters having their bliss point exactly on the frontier of the hyperball Ω_2 are of negligible importance since the measure of the set I is zero.

- Proof.

Let x_2 and $x_2^{'}$ be elements of X such that x_1 x_2 and $x_2^{'}$ belong to a line Δ (as depicted in Figure I). We define α and α' to be the angles formed by Δ and the tangents T and T' to the hyperballs $\Omega_2 = \Omega_2(x_1, \theta_1, x_2, \theta_2)$ and $\Omega_2' = \Omega_2(x_1, \theta_1, x_2', \theta_2)$ passing through x_1. Let t and t' be the intersection points of Δ and the tangents T and T' respectively. The triangles x_1tc_2 and $x_1't'c_2'$ are right angled. Thus we get $\sin\alpha = \frac{r(x_1,\theta_1,x_2,\theta_2)}{\|x_1-x_2\|} = \frac{k}{|1-k^2|} = \sin\alpha'$. The points t and t' are then aligned. This concludes the proof.⧫

It is natural to ask how the model behaves when parties tend toward symmetry, i.e. when $k \to 1$. Recall that $c = \beta x_1 + (1-\beta)x_2$ where $\beta = \frac{1}{1-k^2} < 0$. So $\beta \to +\infty$ as $k \to 1$. Thus, Ω_2 becomes bigger but its center c goes to infinity. In the limit, Ω_2 tends to be the half-space delimited by the median hyperplane[5] between x_1 and x_2 as in the symmetrical model. Again, Figure I provides the geometrical intuition.

Let us now move to the analysis of the game. We define :

$$X_1^* = \{x_1 \in X \,|\forall x_2 \in X,\ S_1(x_1, \theta_1, x_2, \theta_2) \geq 1/2\}$$

to be the set of player's 1 strategies that guarantees him, or her, a score over 1/2.

Lemma 3 X_1^* ***is a closed, convex set.***

First, we prove that X_1^* is a closed set.

- Consider the function $\varphi_{(\theta_1,x_2,\theta_2)} : x_1 \mapsto S_1(x_1, \theta_1, x_2, \theta_2)$. This function is continuous with respect to x_1 for every value of θ_1, x_2 and θ_2. This a direct consequence of the assumption made on the distribution f and the properties of Lebesgue's integral.

- $\varphi^{-1}_{(\theta_1,x_2,\theta_2)}([1/2, 1])$ is then closed because it is the inverse image of a closed set under a continuous function.

- $X_1^* = \cap_{x_2\in X}\varphi^{-1}_{(\theta_1,x_2,\theta_2)}([1/2, 1])$ is thus an intersection of closed sets and so it is a closed set itself.

[5] The median hyperplane of x_1 and x_2 is the set of points in $\mathbb{R}^n$ that are equidistant from x_1 and x_2.

Let us now turn to the convexity problem.

- Consider x and $y \in X_1^*$, a point $z \in [x, y]$ and the mapping $\tilde{c} : x_2 \mapsto c(x_1, \theta_1, x_2, \theta_2)$, which associates with each strategy of player 2 the center of the hyperball $\Omega_2(x_1, \theta_1, x_2, \theta_2)$. Note that the mapping $\tilde{c}$ from X to X is one to one. Consider a hyperball Ω_2 of center $c \in X$. The radius of this hyperball is uniquely determined by the rule of the game. x and $y \in X_1^*$, thus there exists no x_2 such that $S_1(x, \theta_1, x_2, \theta_2) \geq 1/2$ or $S_1(y, \theta_1, x_2, \theta_2) \geq 1/2$. Consider a fixed point c. We study the function $r : z \mapsto r(z)$ associating to every $z \in [x, y]$ the length of the radius of $\Omega_2(z, \theta_1, x_2(z), \theta_2)$ where $x_2(z)$ is chosen so that the center c of $\Omega_2(z, \theta_1, x_2, \theta_2)$ remains fixed when z moves. The function r reaches its maximum either in x or in y when z varies from x to y. Thus, the following equation is satisfied by every $z \in [x, y]$.

  $$S_2(z, \theta_1, x_2(z), \theta_2) \leq \max\left[S_2(x, \theta_1, x_2(x), \theta_2), S_2(y, \theta_1, x_2(y), \theta_2)\right] \leq 1/2$$

  Then there exists no c in X such that the hyperball of center c guarantees player 2 a score greater than 1/2. Since the mapping c is one to one, for any $z \in [x, y]$ player's 2 best reply gives him/her a score strictly less than 1/2. So x and $y \in X_1^*$, implies that any $z \in [x, y]$ also belongs to X_1^*, which proves that X_1^* is convex.♦

We now turn to our main results:

Theorem 4 ***Existence and uniqueness of the equilibrium.***

For every game, one of the following two propositions hold:

- **The set X_1^* is not empty and there exists a unique Nash equilibrium which is a dominant strategy equilibrium.**
- **The set X_1^* is empty and there exists no Nash equilibrium in pure strategy.**
- Proof.

Suppose X_1^* is not empty. Player $1's$ optimal strategy is to choose x_1 in X_1^* minimizing the distance between x_1 and his/her preferred platform p_1. Thanks to the convexity property of the set X_1^* proved in our previous lemma, such a minimum exists and is unique. $x_1^* = \arg\min_{x_1 \in X_1^*} \|x_1 - p_1\|$. The function $S_2(x_1^*, \theta_1, x_2, \theta_2)$ is continuous with respect to x_2, thus it attains its maximum over the compact set X. So, their exists a best reply $x_2^* = \arg\max_{x_2 \in X} S_2(x_1^*, \theta_1, x_2, \theta_2)$. This conclude the proof of the theorem[6]♦

Corollary 5 *If X_1^* is not empty, either $p_1 \in X_1^*$ and 1 wins the election by a score above 1/2 or $p_1 \notin X_1^*$ and player 1 wins the election by a score of 1/2.*

- *Proof*

The proof follows from theorem 1. Either $p_1 \in X_1^*$ and $1's$ optimal strategy is to choose $x_1^* = p_1$. Or $p_1 \notin X_1^*$ and then $x_1^* = \arg\min_{x_1 \in X_1^*} \|x_1 - p_1\|$ lies on the boundary of X_1^* implying a score of 1/2.

Theorem 6 ***For every distribution** f, there exists a unique number $k^*(f)$ **such that the game** (p_1, k, f) **possesses a unique Nash equilibrium in pure strategies whenever** $k \leq k^*(f)$, **and no equilibrium in pure strategy whenever** $k > k^*(f)$.*

- Proof
- Let us prove that every strategy of player 1 belongs to X_1^* when $k \to +\infty$. In order to do so, consider the following expression : $\int_{\Omega_2(x_1,\theta_1,x_2,\theta_2)} df \leq \max_{x \in X} f(x) \times \frac{n}{(n-1)} \pi r(x_1, \theta_1, x_2, \theta_2)^n$ and recall that $r(x_1, \theta_1, x_2, \theta_2) = \frac{k}{|1-k^2|} \|x_1 - x_2\|$ and that X is a bounded set. Thus whenever $k \to +\infty$, $r(x_1, \theta_1, x_2, \theta_2)^n \to 0$ and so, $S_2(x_1, \theta_1, x_2, \theta_2) \to 0$.
- Suppose now that k tends to one. Then the game tends to the symmetric game. In such case X_1^* shrinks to the Condorcet winner when it exists, and is empty otherwise.
- The last step consists of proving that $X_1^*(k)$ is strictly included in $X_1^*(k')$ if and only if $k < k'$. Since $\Omega_2(x_1, \theta_1', x_2, \theta_2') \subset \Omega_2(x_1, \theta_1, x_2, \theta_2)$ if and only if $\frac{\theta_2'}{\theta_1'} < \frac{\theta_2}{\theta_1}$ smaller values of k induce higher scores for player 1. This concludes the proof of the last step.♦

$k^*(f)$ represents a bound above which the set X_1^* is not empty.

[6]The best reply of player 2 may not be uniquely defined. Since this does not affect the outcome of the game, we maintain the assertion that the equilibrium is unique. It is possible to be more precise by assuming that player 2 selects the platform that is closer to his ideal one p_2.

4.1 Computing $k^*(f)$.

Corollary 7 $k^*(f) = 1$ ***if and only if the distribution*** f ***admits a Condorcet winner***.

- Proof.

The proof is obvious and can be omitted⧫

This corollary proves that an equilibrium does not always exist in our general framework but existence is guaranteed in a broader range of cases than in the symmetrical model. In that sense, our model is a real generalization of the symmetrical model. Getting an explicit form for $k^*(f)$ remains an open question we wish to address in further work. Intuition suggests that $k^*(f)$ is a sort of symmetry index of f. The more f is symmetrical around a point the bigger is X_1^*.

5 Conclusion

As mentioned, further research may deepen our understanding of the logic of the asymmetric model. Another line of research is to extend the model to a dynamic model where the θ parameters are fixed endogenously. More generally, our model yields a fresh look at the existing literature on spatial models of voting.

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5.1 Appendix.

Proof of lemma 1.

Let us simplify the expression of r.

$$\begin{aligned}
r^2 &= \frac{k^2\left\|x_2\right\|^2-\left\|x_1\right\|^2}{1-k^2}+\left\|\frac{x_1}{1-k^2}\right\|^2-2\left\langle\frac{x_1}{1-k^2},\frac{k^2x_2}{1-k^2}\right\rangle+\left\|\frac{k^2x_2}{1-k^2}\right\|^2 \\
&= \frac{k^2\left\|x_2\right\|^2-\left\|x_1\right\|^2}{1-k^2}+\frac{\left\|x_1\right\|^2}{\left(1-k^2\right)^2}+\frac{k^4\left\|x_2\right\|^2}{\left(1-k^2\right)^2}-2\left\langle\frac{x_1}{1-k^2},\frac{k^2x_2}{1-k^2}\right\rangle \\
&= \frac{1-(1-k^2)}{\left(1-k^2\right)^2}\left\|x_1\right\|^2+\frac{k^2(1-k^2)+k^4}{\left(1-k^2\right)^2}\left\|x_2\right\|^2-2\frac{k^2}{\left(1-k^2\right)^2}\left\langle x_1,x_2\right\rangle \\
&= \frac{k^2}{\left(1-k^2\right)^2}\left\|x_1\right\|^2+\frac{k^2}{\left(1-k^2\right)^2}\left\|x_2\right\|^2-2\frac{k^2}{\left(1-k^2\right)^2}\left\langle x_1,x_2\right\rangle \\
&= \frac{k^2}{\left(1-k^2\right)^2}\left[\left\|x_1\right\|^2+\left\|x_2\right\|^2-2\left\langle x_1,x_2\right\rangle\right] \\
&= \frac{k^2}{\left(1-k^2\right)^2}\left\|x_{1-}x_2\right\|^2>0
\end{aligned}$$

Finally, we get $r=\frac{k}{|1-k^2|}\left\|x_1-x_2\right\|=\frac{k}{k^2-1}\left\|x_1-x_2\right\|$, since $k>1$.