



Local-Stochastic Volatility : The Dynamics of the Forward Smile

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Local-Stochastic Volatility The Dynamics of the Forward Smile

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Introduction

This talk is broken into two parts.

Part I We will motivate the audience how we expect the existence of the smile to affect pricing of derivatives contracts (due to the speaker's bias, specifically those popular within the FX markets) and consider the two cornerstones of volatility modelling: the local volatility model and a stochastic volatility model. Motivated by macro market dynamics we shall consider a local stochastic volatility model.

- Local volatility model: calculating the local volatility surface and model implementation
- Stochastic volatility model: ADI scheme and model implementation
- Local-stochastic volatility model.

We can very much choose to leave things here and go forth and implement models for pricing and risk management.

Part II Pretty soon we need to address deeper questions in terms of how our model evolves through time and how it constructs prices - and be able to describe this in an intuitive manner. To that end we will explore our stochastic volatility model analytically and be able to investigate the forward smile.

- Review of the Hull-White approach for stochastic volatility
- Constructing the volatility distribution
- Constructing the forward spot-volatility density and generating the forward smile
- Using the forward smile to price hurdle options and forward vol agreements

Part I

Market, Models and Methods

Typical Market Volatility Surface

This is a typical FX volatility surface, in this case corresponding to EURUSD.

EURUSD	BID	ASK	25RR	25FLY	10RR	10FLY
1W	22.41%	23.61%	-1.26%	0.67%	-2.16%	2.23%
2W	22.73%	23.55%	-1.34%	0.70%	-2.34%	2.27%
3W	22.68%	23.37%	-1.40%	0.79%	-2.44%	2.56%
1M	22.24%	22.59%	-1.45%	0.72%	-2.56%	2.66%
2M	22.00%	22.42%	-1.39%	0.75%	-2.51%	2.89%
3M	21.80%	22.15%	-1.36%	0.83%	-2.43%	3.17%
4M	20.96%	21.30%	-1.32%	0.85%	-2.39%	3.25%
6M	19.85%	20.18%	-1.27%	0.88%	-2.30%	3.39%
9M	18.88%	19.19%	-1.22%	0.91%	-2.24%	3.50%
1Y	18.24%	18.52%	-1.18%	0.94%	-2.16%	3.59%
18M	17.82%	18.22%	-1.10%	1.00%	-2.03%	3.71%
2Y	17.54%	17.94%	-1.02%	0.87%	-1.93%	3.40%
5Y	14.27%	14.92%	-0.69%	0.63%	-1.37%	2.49%
10Y	11.24%	11.79%	-0.58%	0.41%	-1.57%	1.66%

The FX volatility surface is constructed as above to indicate the level of 'ATM' volatility (indicated by the Bid/Ask rates) the degree of curvature at the 25 delta and 10 delta region (indicated by the 25FLY and 10FLY rates respectively) and the degree of curvature at the 25 delta and 10 delta region (indicated by the 25RR and 10RR rates respectively).

There are many details that need be considered due to currency pair specific conventions which is outside the scope of this course.

Generating a simple smile

A smile can be generated from this very, very simple two state model. Suppose we toss a coin with outcomes: heads - volatility is $\bar{\sigma} + \Delta\sigma$; tails - volatility is $\bar{\sigma} - \Delta\sigma$.

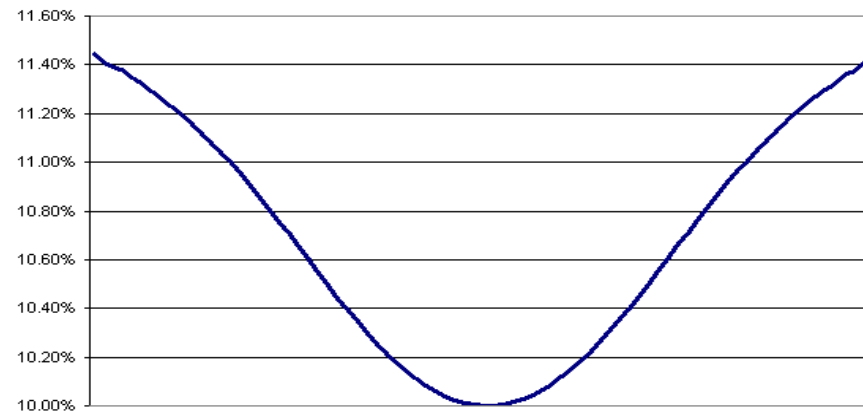
We can write the value of a contract in this world as

$$V(2 - state) = \frac{1}{2} [V(\bar{\sigma} - \Delta\sigma) + V(\bar{\sigma} + \Delta\sigma)] = V(\bar{\sigma}) + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Big|_{\bar{\sigma}} (\Delta\sigma)^2$$

So contracts with positive "volgamma" ie

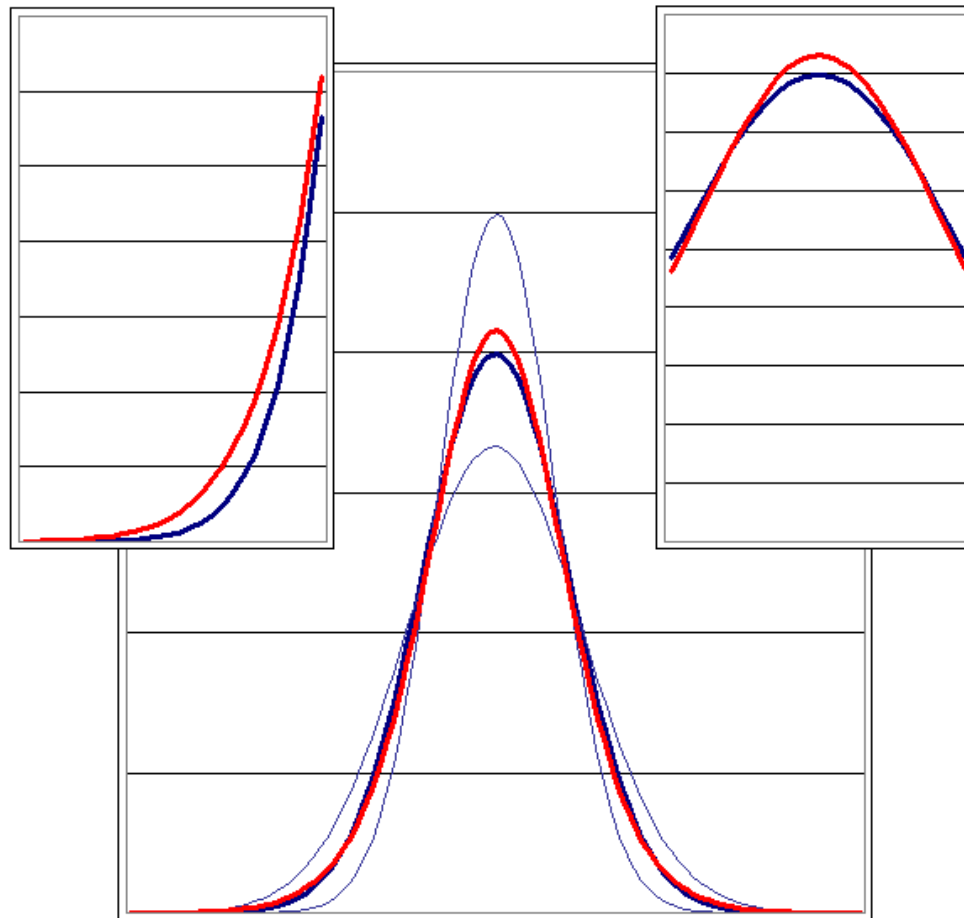
$$\frac{\partial^2 V}{\partial \sigma^2} > 0$$

will trade at a premium to the value of the contract with volatility $\bar{\sigma}$. Suppose $\bar{\sigma} = 10\%$ and $\Delta\sigma = 2\%$ then for a typical expiry the implied volatility across strike will be of the form



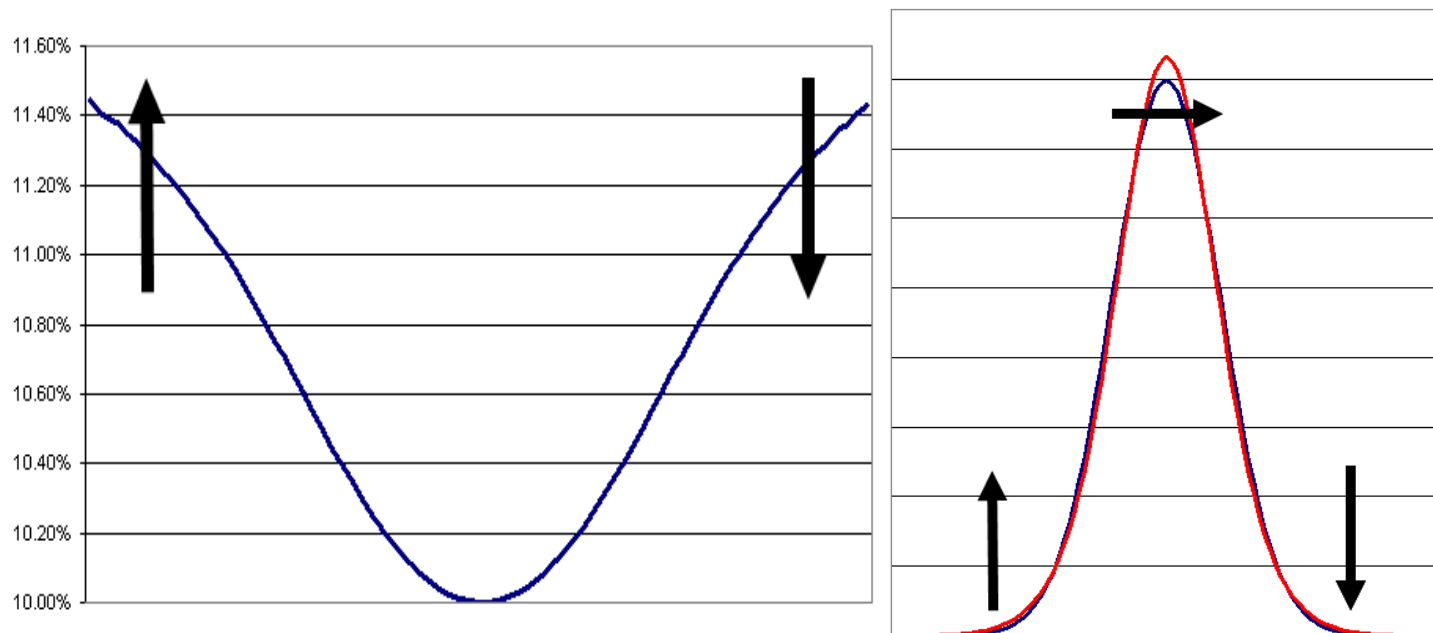
Effect on the underlying probability density function

The resulting density has two profound features 1) thicker tails 2) higher peak.



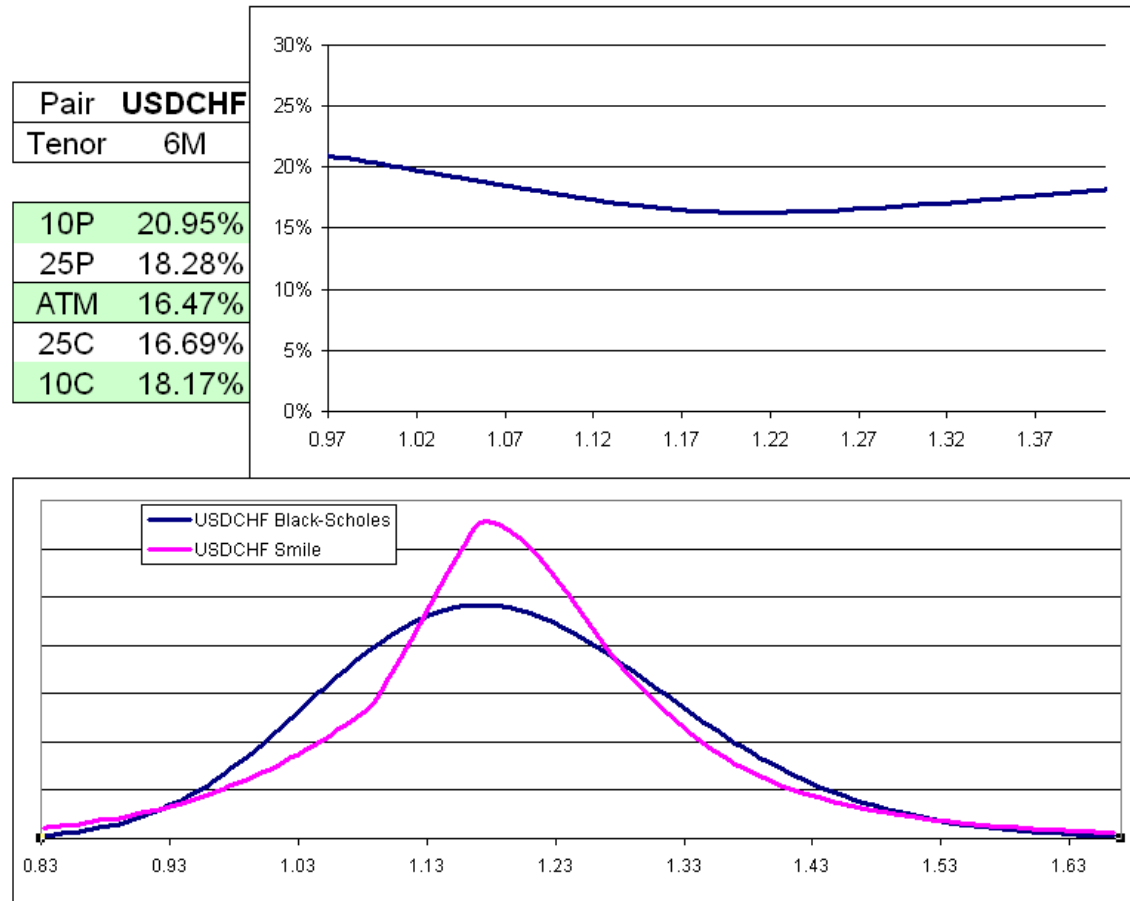
Introducing skew

A skew (risk reversal) can be induced by lowering the smile on one side and raising on the other. This will have the effect of thinning one tail in the implied pdf and fattening the other. However, the density must preserve the expected value of the underlying (equal to the market forward) and so the peak of the distribution needs to move away from the thickening tail:

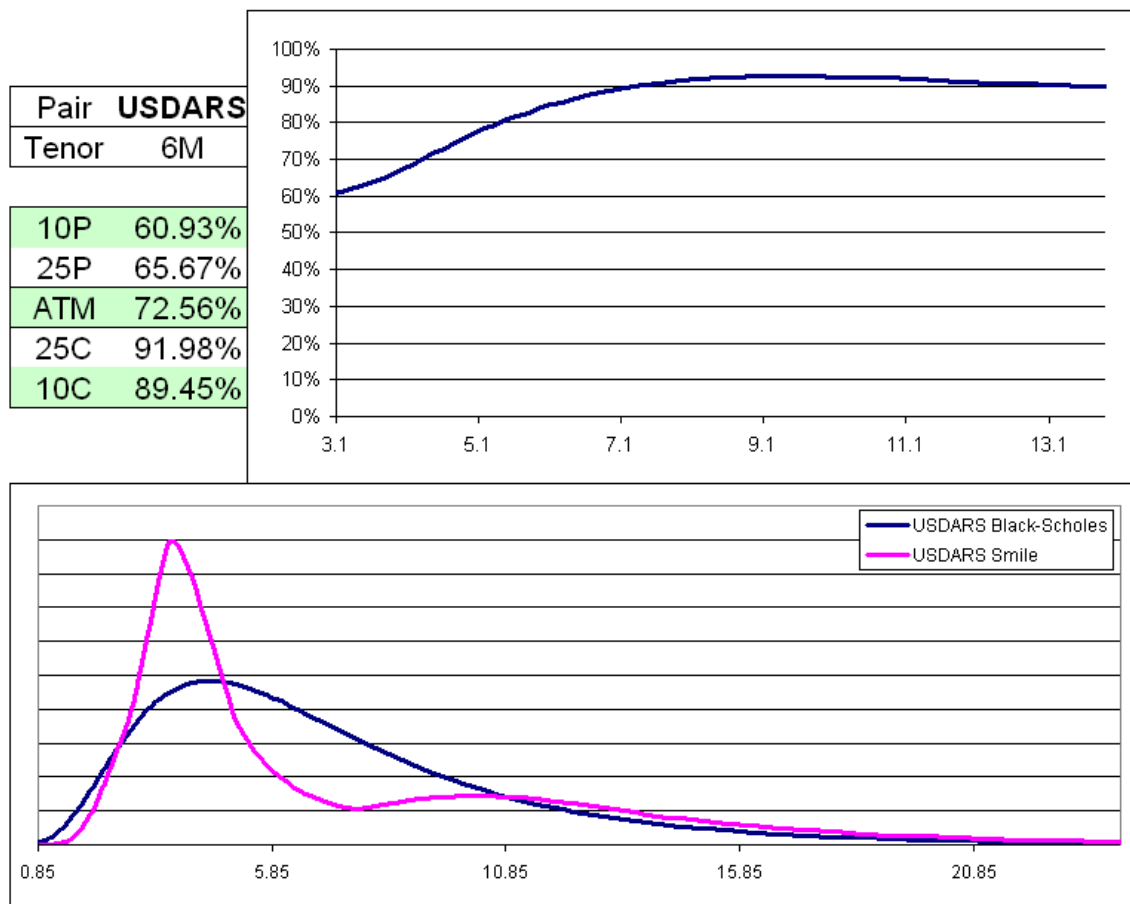


These qualitative features should be observable in real market smiles.

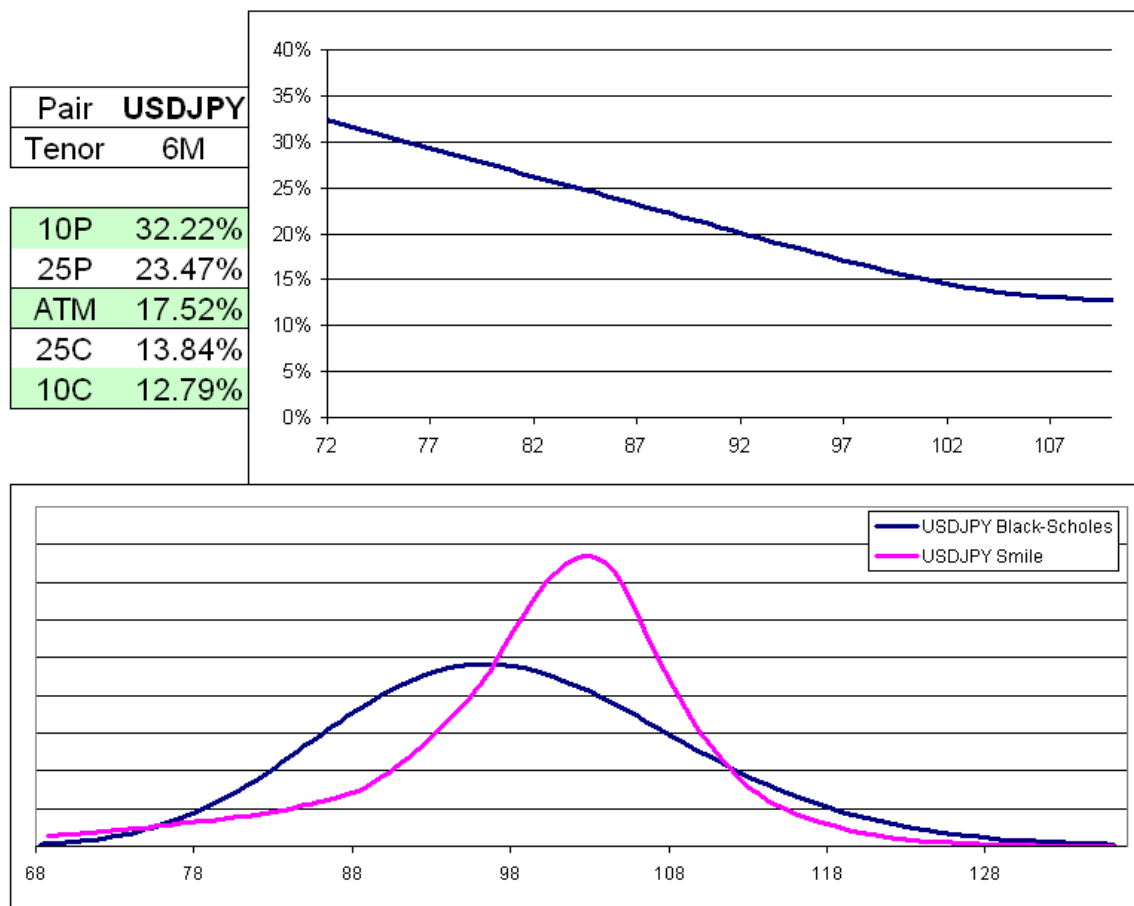
Smile and implied probability density function for USDCHF



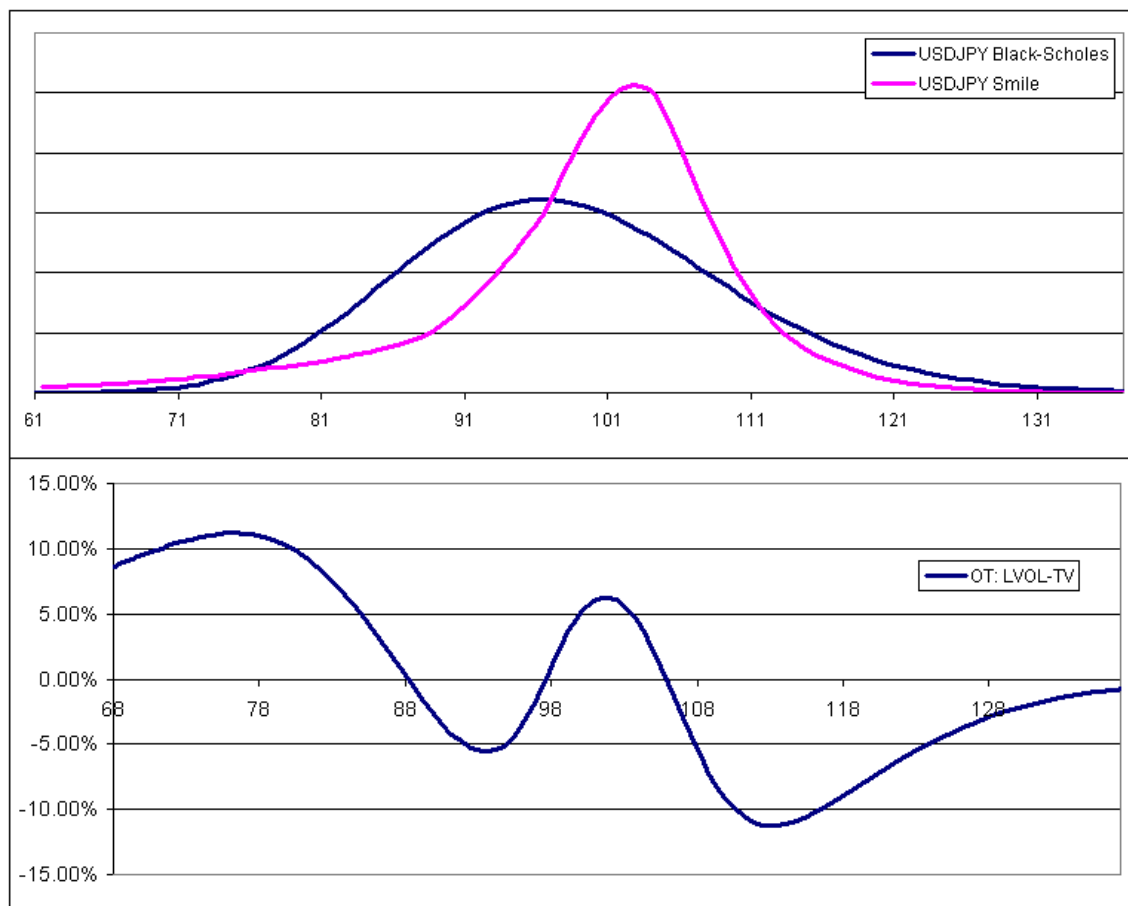
Smile and implied probability density function for USDARS



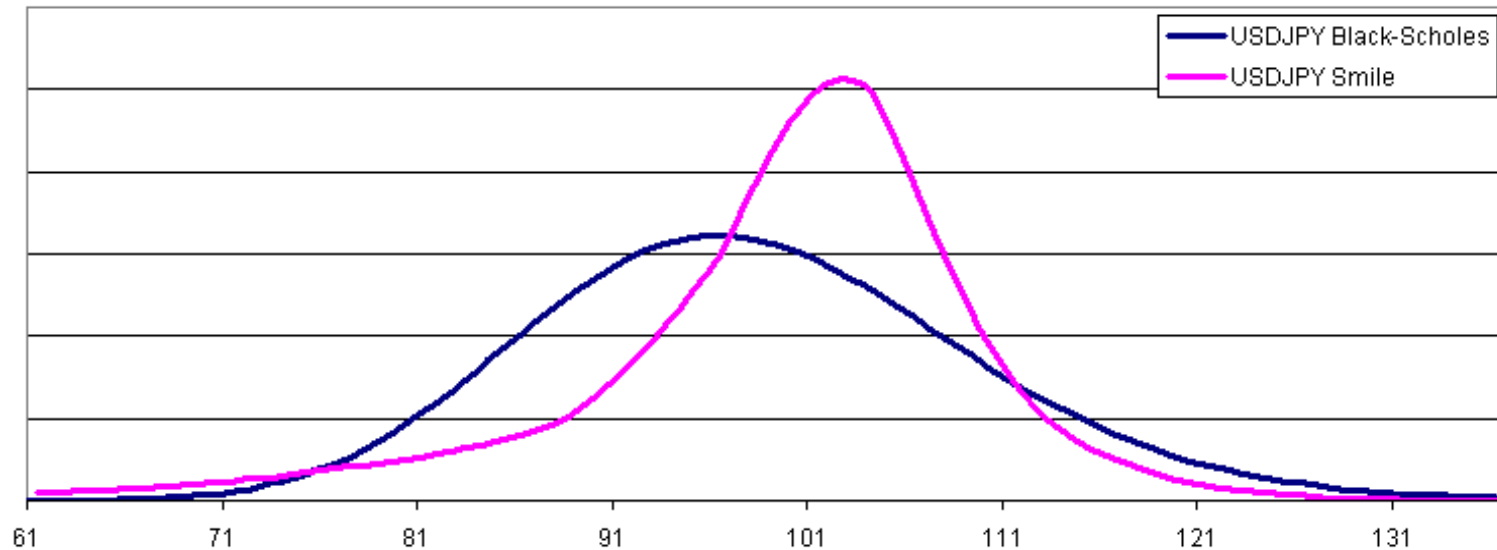
Smile and implied probability density function for USDJPY



From the implied pdf we can gain some qualitative feel for where exotics will trade relative to Black-Scholes. In this case one touch options are considered (OT) - contracts which pay a fixed notional amount of spot ever trades beyond the OT level.

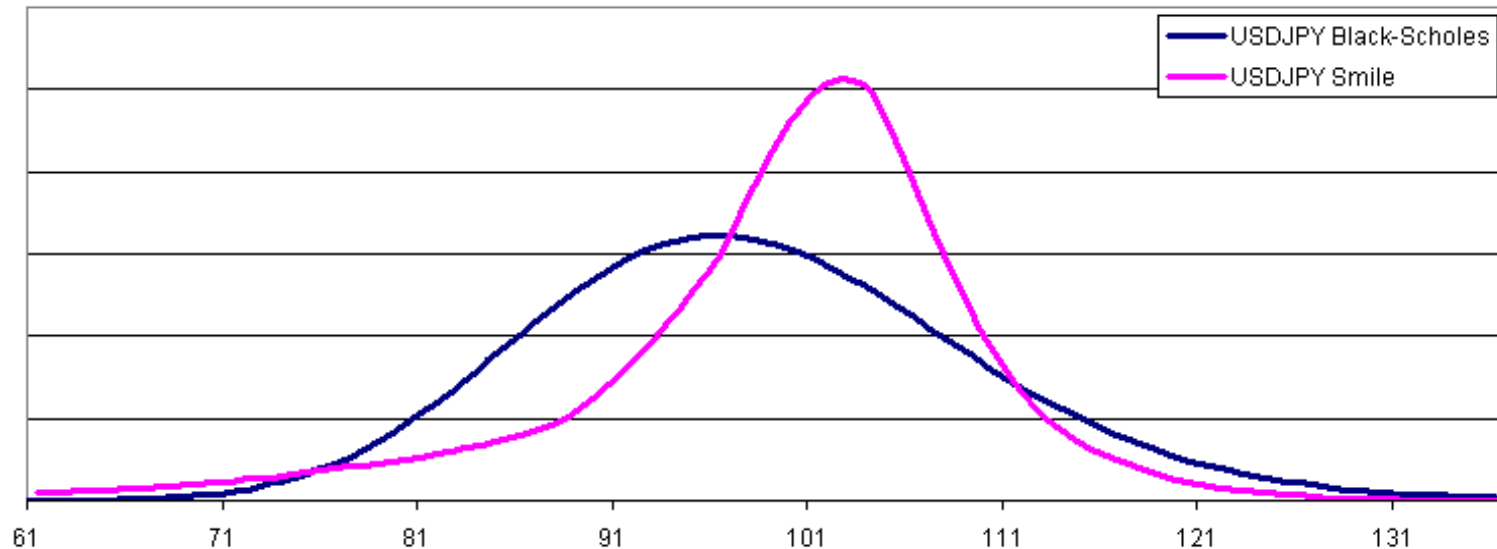


The smile effect on the price of double no touch options. Double no touch options pay a fixed notional amount provided spot remains between the upper and lower levels during the life of the contract. The grid below shows the smile adjustment across a range of upper and lower barrier levels.



	100	103	106	109	112	115	118	121	124	127	130	133	136
95	0.0%	0.0%	1.3%	5.7%	9.3%	10.5%	10.1%	9.1%	8.1%	7.2%	6.5%	5.9%	5.6%
92	0.0%	0.1%	3.7%	10.2%	14.0%	14.4%	13.0%	11.2%	9.5%	8.1%	7.0%	6.3%	5.8%
89	-0.2%	-1.1%	2.6%	9.0%	12.2%	12.0%	10.2%	8.0%	6.1%	4.6%	3.5%	2.7%	2.2%
86	-1.7%	-4.8%	-1.5%	4.8%	7.8%	7.5%	5.5%	3.3%	1.4%	-0.1%	-1.3%	-2.1%	-2.6%
83	-4.5%	-9.3%	-5.9%	0.6%	3.7%	3.3%	1.4%	-0.8%	-2.7%	-4.2%	-5.4%	-6.2%	-6.7%
80	-7.2%	-12.7%	-8.9%	-2.2%	1.0%	0.7%	-1.2%	-3.4%	-5.3%	-6.8%	-8.0%	-8.8%	-9.3%
77	-9.0%	-14.4%	-10.2%	-3.3%	0.0%	-0.3%	-2.2%	-4.4%	-6.3%	-7.8%	-8.9%	-9.7%	-10.3%
74	-9.8%	-14.7%	-10.1%	-3.1%	0.2%	0.0%	-1.9%	-4.1%	-6.0%	-7.5%	-8.7%	-9.5%	-10.0%
71	-9.9%	-14.2%	-9.2%	-2.1%	1.2%	1.0%	-0.9%	-3.1%	-5.0%	-6.5%	-7.6%	-8.4%	-9.0%
68	-9.6%	-13.3%	-8.0%	-0.8%	2.6%	2.3%	0.4%	-1.7%	-3.7%	-5.2%	-6.3%	-7.1%	-7.7%

The smile effect on the price of up-and-out put options. Up-and-out put options are FX vanilla put options which knockout is spot ever breaches the upper knockout level. The grid below shows the smile adjustment across a range of upper barrier levels and put strikes.

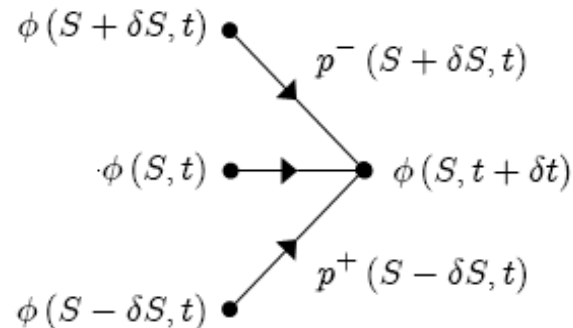


	100	101	102	103	104	105	106	107	108	109	110	111	112
95	0.4%	0.5%	0.6%	0.7%	0.7%	0.7%	0.7%	0.7%	0.6%	0.6%	0.6%	0.5%	0.5%
93	0.5%	0.7%	0.8%	0.9%	0.9%	0.9%	0.9%	0.9%	0.8%	0.8%	0.8%	0.8%	0.8%
91	0.6%	0.8%	0.9%	1.0%	1.0%	1.1%	1.0%	1.0%	1.0%	1.0%	1.0%	1.0%	1.0%
89	0.7%	0.9%	1.0%	1.1%	1.1%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%	1.1%
87	0.7%	0.9%	1.1%	1.2%	1.2%	1.2%	1.3%	1.3%	1.3%	1.3%	1.2%	1.2%	1.2%
85	0.7%	0.9%	1.1%	1.2%	1.2%	1.3%	1.3%	1.3%	1.3%	1.3%	1.3%	1.3%	1.3%
83	0.8%	1.0%	1.1%	1.2%	1.2%	1.3%	1.3%	1.3%	1.3%	1.3%	1.3%	1.3%	1.3%
81	0.7%	0.9%	1.1%	1.2%	1.2%	1.2%	1.3%	1.3%	1.3%	1.3%	1.3%	1.3%	1.3%
79	0.7%	0.9%	1.0%	1.1%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%
77	0.7%	0.9%	1.0%	1.1%	1.1%	1.1%	1.1%	1.1%	1.1%	1.1%	1.1%	1.1%	1.1%

Given we have a qualitative feel for model pricing, it is now time to build a model

Fokker-Plank equation

Here we aim to describe the forward evolution of the probability density function. We express the value of the density at the future time in terms of its values at the previous time and the transitional densities as shown below:



Equating both sides

$$\phi(S, t + \delta t) = \phi(S + \delta S, t) p^-(S + \delta S, t) + \phi(S, t) (1 - p^-(S, t) - p^+(S, t)) + \phi(S - \delta S, t) p^+(S - \delta S, t)$$

and expanding to orders δt and δS^2 then (ignoring higher order terms)

$$\phi(S, t + \delta t) \approx \phi(S, t) + \phi_t(S, t) \delta t$$

$$\phi(S \pm \delta S, t) \approx \phi(S, t) \pm \phi_S(S, t) \delta S \pm \frac{1}{2} \phi_{SS}(S, t) \delta S^2$$

$$p^\pm(S \mp \delta S, t) \approx p^\pm(S, t) \mp p_S^\pm(S, t) \delta S \pm \frac{1}{2} p_{SS}^\pm(S, t) \delta S^2$$

then

$$\phi_t(S, t) \delta t = [\phi(S, t) (p^+(S, t) \delta S - p^-(S, t) \delta S)]_S + \frac{1}{2} [\phi(S, t) (p^+(S, t) \delta S^2 + p^-(S, t) \delta S^2)]_{SS}$$

which can be re-expressed

$$\phi_t(S, t) \delta t = [\phi(S, t) E[\delta S]]_S + \frac{1}{2} [\phi(S, t) E[\delta S^2]]_{SS}$$

Local Volatility

For the process

$$\frac{dS}{S} = \mu dt + \sigma(S, t) dW$$

then

$$\phi_t(S, t) = [\phi(S, t) \mu S]_S + \frac{1}{2} [\phi(S, t) \sigma(S, t)^2 S^2]_{SS}$$

From the price of a call option then differentiating with respect to t

$$C_t(K, t) = -r_d C + DF \int_K^\infty (S - K) \phi_t(S, t) dS$$

Substituting the Fokker-Plank equation and using the known strike derivatives of C results in

$$C_t(K, t) = -r_f C(K, t) + \mu K C_K(K, t) + \frac{1}{2} C_{KK}(K, t) \sigma(K, t)^2 K^2$$

so rearranging

$$\sigma(K, t) = \sqrt{\frac{2(C_t(K, t) + r_f C(K, t) - \mu K C_K(K, t))}{C_{KK}(K, t) K^2}}$$

By construction local volatility will ensure that any pricing is consistent with the volatility surface and so it is necessary to solve the pricing problem

$$\frac{\partial V}{\partial t} + \sigma(S, t)^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - r_d V = 0$$

subject to the appropriate boundary conditions.

The Crank-Nicholson Scheme

There are a number of finite difference methods available to solve one factor problems such as pricing under local vol: explicit, implicit, Crank-Nicholson, Douglas, etc. Of interest to us is an equation of the form

$$\frac{\partial V}{\partial t} + a(S, t) \frac{\partial^2 V}{\partial S^2} + b(S, t) \frac{\partial V}{\partial S} + c(S, t) V = 0$$

We construct a grid of $N + 1$ time points and $M + 1$ spot points. $V_{j,k}$ corresponds to the j^{th} spot index ($0 \leq j \leq M$) and k^{th} time index ($0 \leq k \leq N$). The initial conditions are set by specifying $V_{j,N}$ ($0 \leq j \leq M$) and there are a number of ways of defining the boundary conditions corresponding to $j = 0, M$ for $0 \leq k < N$ either as value conditions (Dirichelet), derivative (Von Neumann), zero curvative conditions, etc. Within the Crank-Nicholson scheme for interior points discretization is as follows

$$\begin{aligned} \frac{\partial V}{\partial t} &\simeq \frac{V_{j,k+1} - V_{j,k}}{\Delta T} \\ a(S, t) \frac{\partial^2 V}{\partial S^2} &\simeq \frac{1}{2} \left[a_{j,k} \frac{\partial^2 V_{j,k}}{\partial S^2} + a_{j,k+1} \frac{\partial^2 V_{j,k+1}}{\partial S^2} \right] \\ b(S, t) \frac{\partial V}{\partial S} &\simeq \frac{1}{2} \left[b_{j,k} \frac{\partial V_{j,k}}{\partial S} + b_{j,k+1} \frac{\partial V_{j,k+1}}{\partial S} \right] \\ c(S, t) V &\simeq \frac{1}{2} [c_{j,k} V_{j,k} + c_{j,k+1} V_{j,k+1}] \end{aligned}$$

Then rearranging the discretized equation

$$V_{j,k} - \frac{\Delta T}{2} \left[a_{j,k} \frac{\partial^2 V_{j,k}}{\partial S^2} + b_{j,k} \frac{\partial V_{j,k}}{\partial S} + c_{j,k} V_{j,k} \right] = V_{j,k+1} + \frac{\Delta T}{2} \left[a_{j,k+1} \frac{\partial^2 V_{j,k+1}}{\partial S^2} + b_{j,k+1} \frac{\partial V_{j,k+1}}{\partial S} + c_{j,k+1} V_{j,k+1} \right]$$

The derivatives as calculated as

$$\begin{aligned} \frac{\partial V_{j,k}}{\partial S} &= \frac{V_{j+1,k} - V_{j-1,k}}{2\Delta S} \\ \frac{\partial^2 V_{j,k}}{\partial S^2} &= \frac{V_{j+1,k} + V_{j-1,k} - 2V_{j,k}}{\Delta S^2} \end{aligned}$$

So

$$\begin{aligned} & V_{j,k} - \frac{\Delta T}{2} \left[a_{j,k} \frac{\partial^2 V_{j,k}}{\partial S^2} + b_{j,k} \frac{\partial V_{j,k}}{\partial S} + c_{j,k} V_{j,k} \right] \\ = & V_{j,k} - \frac{\Delta T}{2} \left[a_{j,k} \frac{V_{j+1,k} + V_{j-1,k} - 2V_{j,k}}{\Delta S^2} + b_{j,k} \frac{V_{j+1,k} - V_{j-1,k}}{2\Delta S} + c_{j,k} V_{j,k} \right] \\ = & \left(-\frac{1}{2} a_{j,k} \frac{\Delta T}{\Delta S^2} - \frac{1}{4} b_{j,k} \frac{\Delta T}{\Delta S} \right) V_{j+1,k} + \left(1 + \frac{\Delta T}{\Delta S^2} a_{j,k} - \frac{\Delta T}{2} c_{j,k} \right) V_{j,k} + \left(-\frac{1}{2} a_{j,k} \frac{\Delta T}{\Delta S^2} + \frac{1}{4} b_{j,k} \frac{\Delta T}{\Delta S} \right) V_{j-1,k} \end{aligned}$$

The discretized equation can be written in matrix form

$$\mathbf{T} \mathbf{V}_k = \mathbf{V}_{k+1}$$

where \mathbf{T} is a tridiagonal matrix and $\mathbf{V}_k^T = (V_{0,k}, V_{1,k}, \dots, V_{M,k})$, $\mathbf{V}_{k+1}^T = (V_{0,k+1}, V_{1,k+1}, \dots, V_{M,k+1})$ are the solution vectors at slice k and $k + 1$ respectively. Crank-Nicholson is an $O(\Delta T^2, \Delta S^2)$ method. See further reading for more details.

Local-Stochastic Volatility

It can be shown that for a process

$$dS = \mu S dt + \sigma_t S dW$$

where σ_t is the instantaneous volatility at time t that in order to reprice all options on today's volatility surface then

$$E[\sigma_t^2 | S = K] = \sigma_{Local}^2(K, T)$$

So, for example the local stochastic volatility model of the form

$$\begin{aligned} dS &= \mu S dt + \lambda(S, t) \sigma_t dW_1 \\ dX &= \alpha(\bar{X}^* - X) dt + \xi dW_2 \end{aligned}$$

where $X = \ln \sigma_t$ then if

$$\lambda(S, t) = \sqrt{\frac{\sigma_{Local}^2(K, T)}{E[\sigma_t^2 | S = K]}}$$

then the model will reprice today's volatility surface. When $\xi = 0$ the model collapses to that of local volatility. Once the scaling function $\lambda(S, t)$ has been found then a two dimensional finite difference scheme can be used to evaluate contracts. The scaling function itself is found by evolving the joint spot - instantaneous volatility forward in time using the two dimensional Fokker-Plank equation and applying the conditional expectation above to determine $\lambda(S, t)$.

The ADI scheme

Here we are trying to solve the differential equation of the form

$$\frac{\partial V}{\partial t} + a(X, Y, t) \frac{\partial^2 V}{\partial X^2} + b(X, Y, t) \frac{\partial V}{\partial X} + c(X, Y, t) \frac{\partial^2 V}{\partial Y^2} + d(X, Y, t) \frac{\partial V}{\partial Y} + e(X, Y, t) \frac{\partial^2 V}{\partial X \partial Y} + f(X, Y, t) V = 0$$

As in the case of Crank-Nicholson

$$\begin{aligned} \frac{\partial V}{\partial t} &\simeq \frac{V_{i,j}^{k+1} - V_{i,j}^k}{\Delta T} \\ a(X, Y, t) \frac{\partial^2 V}{\partial X^2} &\simeq \frac{1}{2} \left[a_{i,j}^k \frac{\partial^2 V_{i,j}^k}{\partial X^2} + a_{i,j}^{k+1} \frac{\partial^2 V_{i,j}^{k+1}}{\partial X^2} \right] \\ b(X, Y, t) \frac{\partial V}{\partial X} &\simeq \frac{1}{2} \left[b_{i,j}^k \frac{\partial V_{i,j}^k}{\partial X} + b_{i,j}^{k+1} \frac{\partial V_{i,j}^{k+1}}{\partial X} \right] \\ c(X, Y, t) \frac{\partial^2 V}{\partial Y^2} &\simeq \frac{1}{2} \left[c_{i,j}^k \frac{\partial^2 V_{i,j}^k}{\partial Y^2} + c_{i,j}^{k+1} \frac{\partial^2 V_{i,j}^{k+1}}{\partial Y^2} \right] \\ d(X, Y, t) \frac{\partial V}{\partial Y} &\simeq \frac{1}{2} \left[d_{i,j}^k \frac{\partial V_{i,j}^k}{\partial Y} + d_{i,j}^{k+1} \frac{\partial V_{i,j}^{k+1}}{\partial Y} \right] \\ e(X, Y, t) \frac{\partial^2 V}{\partial X \partial Y} &\simeq e_{j,k+1} \frac{\partial^2 V_{i,j}^{k+1}}{\partial X \partial Y} \\ f(S, t) V &\simeq \frac{1}{2} [f_{i,j}^k V_{i,j}^k + f_{i,j}^{k+1} V_{i,j}^{k+1}] \end{aligned}$$

The entire system can then be written in the form

$$(1 - \vartheta_X^k - \vartheta_Y^k) V_{i,j}^k = (1 + \vartheta_X^{k+1} + \vartheta_Y^{k+1}) V_{i,j}^{k+1} + e_{i,j}^{k+1} \frac{\partial^2 V_{i,j}^{k+1}}{\partial X \partial Y}$$

where the operators ϑ_X and ϑ_Y contain the X and Y differential operators. The system is approximated as

$$(1 - \vartheta_X^k)(1 - \vartheta_Y^k) V_{i,j}^k = (1 + \vartheta_X^{k+1})(1 + \vartheta_Y^{k+1}) V_{i,j}^{k+1} + e_{j,k+1} \frac{\partial^2 V_{i,j}^{k+1}}{\partial X \partial Y}$$

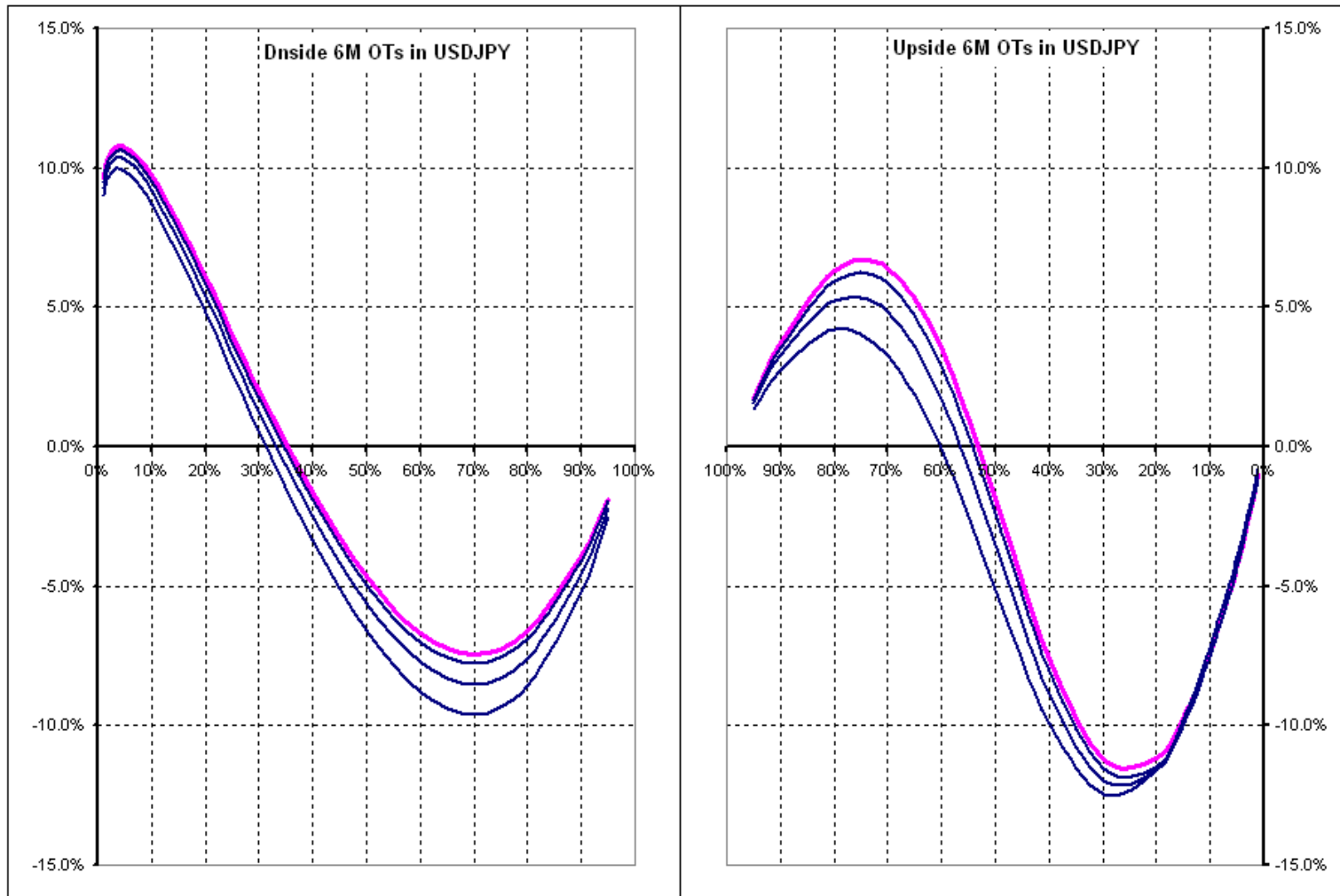
which is split into

$$\begin{aligned} (1 - \vartheta_X^k) \widehat{V}_{i,j}^* &= (1 + \vartheta_X^{k+1} + 2\vartheta_Y^{k+1}) V_{i,j}^{k+1} + e_{j,k+1} \frac{\partial^2 V_{i,j}^{k+1}}{\partial X \partial Y} \\ (1 - \vartheta_Y^k) \widehat{V}_{i,j}^k &= \widehat{V}_{i,j}^* - \vartheta_Y^{k+1} V_{i,j}^{k+1} \end{aligned}$$

Typically a second sweep is performed to improve the accuracy in the presence of the cross term

$$\begin{aligned} (1 - \vartheta_X^k) V_{i,j}^* &= (1 + \vartheta_X^{k+1} + 2\vartheta_Y^{k+1}) V_{i,j}^{k+1} + \frac{1}{2} e_{j,k+1} \left(\frac{\partial^2 V_{i,j}^{k+1}}{\partial X \partial Y} + \frac{\partial^2 \widehat{V}_{i,j}^{k+1}}{\partial X \partial Y} \right) \\ (1 - \vartheta_Y^k) V_{i,j}^k &= V_{i,j}^* - \vartheta_Y^{k+1} V_{i,j}^{k+1} \end{aligned}$$

Effect of introducing stochastic volatility to the local volatility model. Remember, all these exotic prices are by construction consistent with today's volatility surface.



Non-smile risk corresponding to a 6M 105 OT in USDJPY

% change	Spot	PV	PnL	Theta	Delta	Gamma	Vega	Strangle	RiskRev
5.00%	102.56	956,264	289,668	-232	6,083,606	52,846	1,638	0	0
4.00%	101.58	898,100	231,504	-768	6,067,530	95,734	5,242	0	0
3.00%	100.61	839,697	173,101	-1,292	6,015,807	132,081	8,766	0	0
2.00%	99.63	781,406	114,810	-1,795	5,927,769	167,535	12,152	0	0
1.00%	98.65	723,586	56,990	-2,269	5,803,919	200,832	15,342	0	0
0.75%	98.41	709,246	42,651	-2,381	5,767,500	208,746	16,102	0	0
0.50%	98.16	694,965	28,369	-2,492	5,728,948	216,515	16,845	0	0
0.25%	97.92	680,746	14,150	-2,599	5,688,296	224,092	17,572	0	0
0.10%	97.77	672,247	5,651	-2,662	5,662,906	228,520	17,999	0	0
0.00%	97.67	666,596	0	-2,704	5,645,571	231,436	18,280	0	0
-0.10%	97.58	660,956	-5,640	-2,745	5,627,910	234,348	18,558	0	0
-0.25%	97.43	652,520	-14,076	-2,806	5,600,809	238,615	18,969	0	0
-0.50%	97.19	638,523	-28,073	-2,905	5,554,052	245,586	19,638	0	0
-0.75%	96.94	624,612	-41,984	-3,001	5,505,333	252,284	20,287	0	0
-1.00%	96.70	610,790	-55,806	-3,094	5,454,695	258,800	20,915	0	0
-2.00%	95.72	556,511	-110,085	-3,432	5,233,906	282,386	23,203	0	0
-3.00%	94.74	504,079	-162,517	-3,712	4,986,390	301,728	25,108	0	0
-4.00%	93.77	453,792	-212,804	-3,930	4,715,863	316,652	26,602	0	0
-5.00%	92.79	405,914	-260,682	-4,085	4,426,451	326,758	27,670	0	0

Smile adjustment risk corresponding to a 6M 105 OT in USDJPY

% change	Spot	PV	PnL	Theta	Delta	Gamma	Vega	Strangle	RiskRev
5.00%	102.56	14,177	-36,550	220	-3,161,193	-712,900	-1,231	116	-72
4.00%	101.58	40,964	-9,772	602	-2,403,389	-912,806	-3,406	264	-269
3.00%	100.61	59,743	9,016	807	-1,450,864	-1,093,591	-4,423	183	-467
2.00%	99.63	68,517	17,790	740	-320,972	-1,223,847	-3,744	-151	-606
1.00%	98.65	65,668	14,942	328	905,398	-1,232,309	-988	-635	-643
0.75%	98.41	63,049	12,322	172	1,209,987	-1,205,195	28	-754	-636
0.50%	98.16	59,673	8,946	-3	1,508,272	-1,163,365	1,163	-864	-624
0.25%	97.92	55,557	4,830	-194	1,796,598	-1,115,244	2,404	-961	-606
0.10%	97.77	52,743	2,016	-315	1,963,389	-1,073,899	3,194	-1,010	-594
0.00%	97.67	50,727	0	-398	2,071,556	-1,049,654	3,738	-1,039	-584
-0.10%	97.58	48,602	-2,125	-483	2,177,364	-1,021,781	4,294	-1,065	-575
-0.25%	97.43	45,215	-5,511	-613	2,330,514	-969,730	5,148	-1,096	-559
-0.50%	97.19	39,065	-11,662	-836	2,569,824	-885,773	6,616	-1,127	-530
-0.75%	96.94	32,324	-18,403	-1,061	2,786,508	-785,186	8,120	-1,129	-498
-1.00%	96.70	25,049	-25,677	-1,287	2,978,615	-674,800	9,638	-1,101	-463
-2.00%	95.72	-8,005	-58,732	-2,098	3,446,252	-159,624	15,342	-666	-290
-3.00%	94.74	-43,416	-94,142	-2,587	3,373,464	346,868	19,281	235	-29
-4.00%	93.77	-76,042	-126,769	-2,671	2,869,182	653,890	20,796	1,465	368
-5.00%	92.79	-102,341	-153,068	-2,410	2,133,873	771,195	20,091	2,839	894

Part II

Understanding the forward smile

Stochastic Volatility - Hull-White

Let's start with the Hull-White approach. Suppose spot and volatility are uncorrelated. The total variance V is defined by

$$V = \frac{1}{T} \int_0^T \sigma(t)^2 dt$$

and so the option price is given by

$$C(T, K) = \int_0^\infty C(V, T, K) \varphi(V) dV$$

where $\varphi(V)$ is the distribution of the total variance. Here Hull-White perform at Taylor expansion about $\bar{V} = E[V]$ ie

$$C(V, T, K) \equiv C(V) = C(\bar{V}) + \left. \frac{\partial C}{\partial V} \right|_{\bar{V}} (V - \bar{V}) + \frac{1}{2} \left. \frac{\partial^2 C}{\partial V^2} \right|_{\bar{V}} (V - \bar{V})^2 + \dots$$

and use within the integral

$$\begin{aligned} C(T, K) &= \int_0^\infty \left(C(\bar{V}) + \left. \frac{\partial C}{\partial V} \right|_{\bar{V}} (V - \bar{V}) + \frac{1}{2} \left. \frac{\partial^2 C}{\partial V^2} \right|_{\bar{V}} (V - \bar{V})^2 + \dots \right) \varphi(V) dV \\ &= C(\bar{V}) + \frac{1}{2} \left. \frac{\partial^2 C}{\partial V^2} \right|_{\bar{V}} E \left[(V - \bar{V})^2 \right] + \dots \\ &= C(\bar{V}) + \frac{1}{2} \left. \frac{\partial^2 C}{\partial V^2} \right|_{\bar{V}} Var(V) + \dots \end{aligned}$$

Here we need to calculate $E[V]$ and $E[V^2]$ where

$$E[V] = E\left[\frac{1}{T} \int_0^T \sigma(t)^2 dt\right] = \frac{1}{T} \int_0^T E[\sigma(t)^2] dt$$

$$E[V^2] = E\left[\left(\frac{1}{T} \int_0^T \sigma(t)^2 dt\right)^2\right] = \frac{1}{T^2} \int_0^T \int_0^T E[\sigma(t_1)^2 \sigma(t_2)^2] dt_1 dt_2$$

The volatility process used by Hull-White is $d\sigma^2 = \kappa\sigma^2 dt + \xi\sigma^2 dW$ giving

$$E[V] = \frac{e^{\kappa T} - 1}{\kappa T} V_0$$

$$E[V^2] = \left[\frac{2e^{(2\kappa + \xi^2)T}}{(\kappa + \xi^2)(2\kappa + \xi^2)T^2} + \frac{2}{\kappa T^2} \left(\frac{1}{2\kappa + \xi^2} - \frac{e^{\kappa T}}{\kappa + \xi^2} \right) \right] V_0^2$$

For low ξ then the Hull-White approach provides a reasonable approximation, but for larger ξ values needed to calibrate to the market the accuracy deteriorates. We would like to modify the Hull-White process and tackle problem slightly differently, namely

- a mean reverting variance process to better match the shape of the market ATM structure
- In the integral

$$C(T, K) = \int_0^\infty C(V, T, K) \varphi(V) dV$$

work on not $C(V, T, K)$ but $\varphi(V)$ and integrate under our proposed distribution.

Stochastic Volatility - an alternative to Hull-White

Define $X = \ln \sigma^2$. The under Hull-White

$$dX = d \ln \sigma^2 = \frac{1}{\sigma^2} d\sigma^2 - \frac{1}{2\sigma^4} (d\sigma^2)^2 = \kappa dt + \xi dW - \frac{1}{2\sigma^4} \xi^2 \sigma^4 dt = \left(\kappa - \frac{1}{2} \xi^2 \right) dt + \xi dW$$

We would much prefer a mean-reverting volatility process so we propose

$$dX = \alpha (X^* - X) dt - \frac{1}{2} \xi^2 dt + \xi dW = \alpha (\bar{X}^* - X) dt + \xi dW$$

where

$$\bar{X}^* = X^* - \frac{1}{2\alpha} \xi^2$$

Further in the expression

$$C(T, K) = \int_0^\infty C(V, T, K) \varphi(V) dV$$

Hull-White choose to expand on $C(V, T, K)$. Here, instead, we focus attention on $\varphi(V)$ and approximate initially as a lognormal. In order to calculate $E[V]$ and $E[V^2]$ we write $Y = X e^{\alpha t}$ so

$$\begin{aligned} dY &= e^{\alpha t} dX + \alpha X e^{\alpha t} dt \\ &= \alpha e^{\alpha t} (\bar{X}^* - X) dt + \xi e^{\alpha t} dW + \alpha X e^{\alpha t} dt \\ &= \alpha e^{\alpha t} \bar{X}^* dt + \xi e^{\alpha t} dW \end{aligned}$$

so that

$$\begin{aligned} \int_0^T dY &= \int_0^T \alpha e^{\alpha t} \bar{X}^* dt + \int_0^T \xi e^{\alpha t} dW \\ Y(T) - Y(0) &= (e^{\alpha T} - 1) \bar{X}^* + \int_0^T \xi e^{\alpha t} dW \\ X(T) &= X(0) + \bar{X}^* (1 - e^{-\alpha T}) + \frac{\xi}{\sqrt{2\alpha T}} (1 - e^{-2\alpha T})^{1/2} \bar{W}_T \end{aligned}$$

where \bar{W}_T is normal with mean zero and variance T .

Calculating $E[V]$, $E[V^2]$

$$\begin{aligned} E[V] &= E \left[\frac{1}{T} \int_0^T \sigma(t)^2 dt \right] \\ &= \frac{1}{T} \int_0^T E \left[\sigma(t)^2 \right] dt \\ &= \frac{1}{T} \int_0^T \exp \left(X_0 e^{-\alpha t} + X^* (1 - e^{-\alpha t}) + \frac{1}{2} \frac{\xi^2}{2\alpha t} (1 - e^{-2\alpha t}) \right) dt \end{aligned}$$

and secondly

$$E[V^2] = E \left[\frac{1}{T^2} \int_0^T dt_1 \int_0^T dt_2 \sigma(t_1)^2 \sigma(t_2)^2 \right] = \frac{1}{T^2} \int_0^T dt_1 \int_0^T dt_2 E \left[\sigma(t_1)^2 \sigma(t_2)^2 \right]$$

It can be shown that

$$E \left[\sigma(t_1)^2 \sigma(t_2)^2 \right] = E \left[\sigma(t_1)^2 \right] E \left[\sigma(t_2)^2 \right] \exp \left(\frac{\xi^2}{2\alpha} (1 - e^{-2\alpha t_1}) e^{-\alpha(t_2-t_1)} \right)$$

allowing for $E[V^2]$ to be calculated at least numerically.

If V is lognormal then it has the form

$$\varphi(V) = \frac{1}{\sqrt{2\pi\beta}} \exp \left(-\frac{1}{2\beta} (\ln V - \alpha)^2 \right)$$

with

$$\begin{aligned} E[V] &= \exp \left(\alpha + \frac{1}{2}\beta \right) \\ E[V^2] &= E[V]^2 \exp(\beta) \end{aligned}$$

So finally

$$C(T, K) = \frac{1}{\sqrt{2\pi\beta}} \int_0^\infty C(V, T, K) \exp \left(-\frac{1}{2\beta} (\ln V - \alpha)^2 \right) dV$$

In practice, the assumption of lognormality for the distribution $\varphi(V)$ is a good one but can be improved by considering higher order moments $E[V^3]$, $E[V^4]$,... . There are a variety of ways in which to include higher order moments, for example the Gram-Charlier or Edgeworth series.

A Gram-Charlier series would take the form

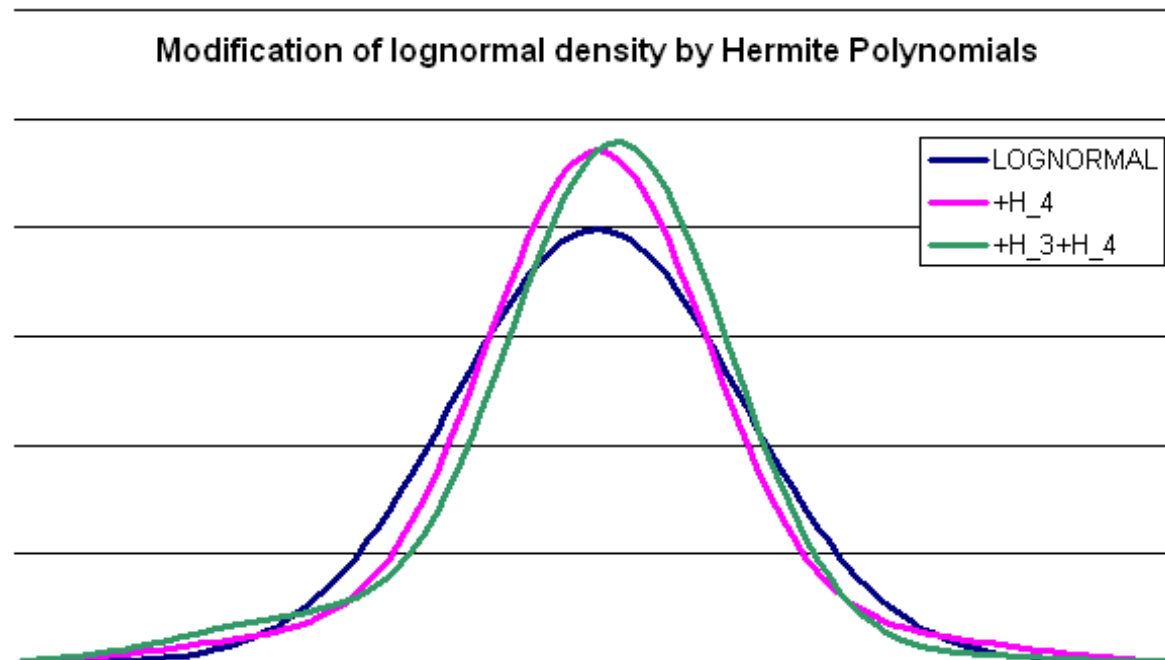
$$\varphi(V) = \frac{1}{\sqrt{2\pi\beta}} \exp\left(-\frac{1}{2\beta}(\ln V - \alpha)^2\right) \left[1 + \frac{\kappa_3}{3!\beta\sqrt{\beta}} H_3\left(\frac{\ln V - \alpha}{\sqrt{\beta}}\right) + \frac{\kappa_4}{4!\beta^2} H_4\left(\frac{\ln V - \alpha}{\sqrt{\beta}}\right) + \dots\right]$$

where

$$H_3(x) = x^3 - 3x$$

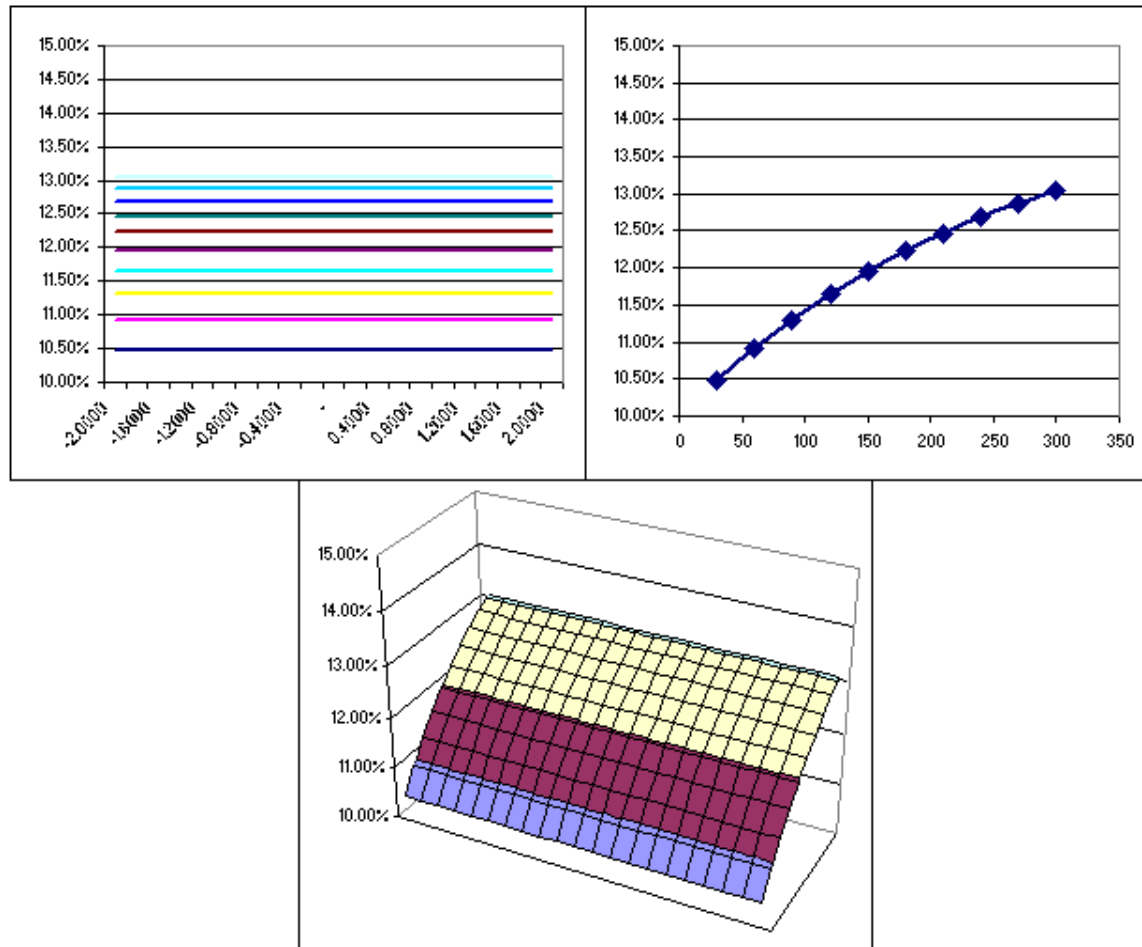
$$H_4(x) = x^4 - 6x^2 + 3$$

and κ_3 and κ_4 allow us to incorporate skewness and kurtosis respectively.

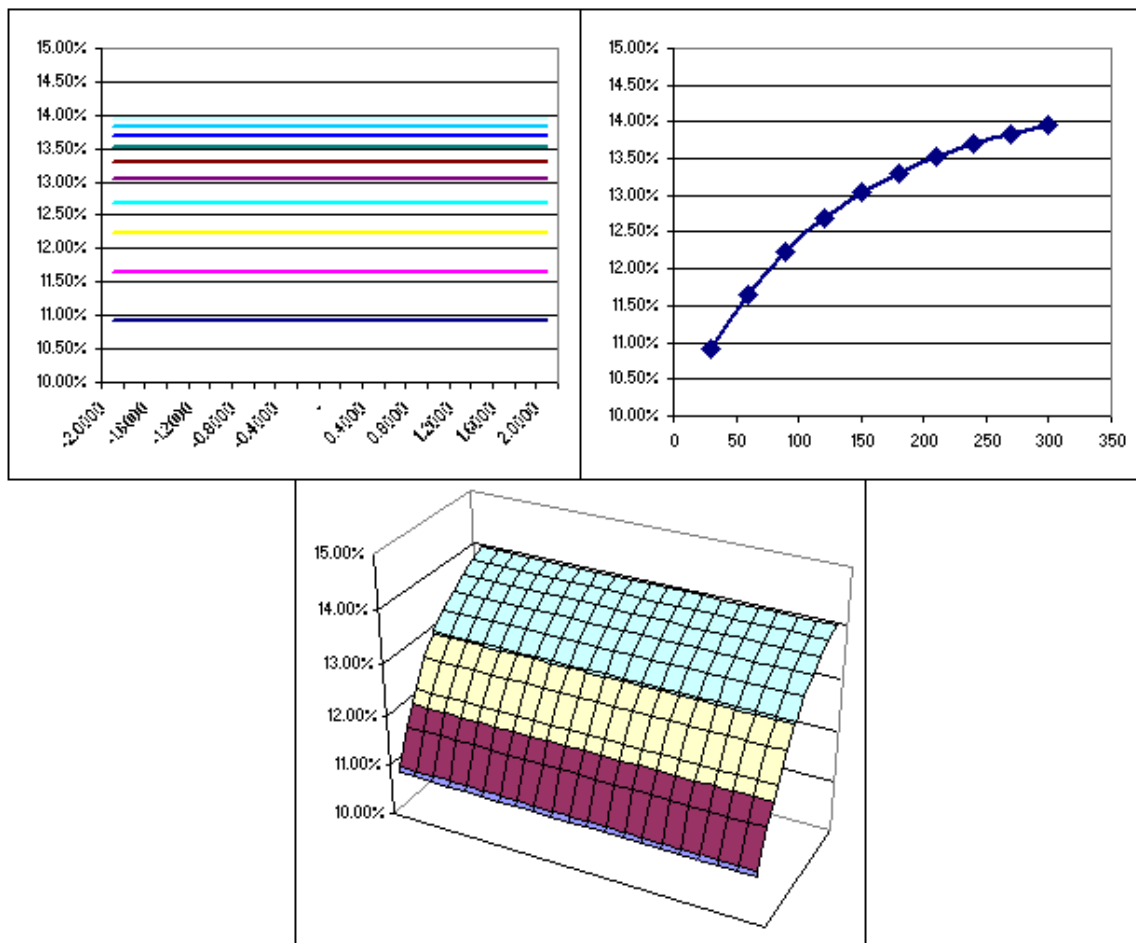


Armed with a relatively simple method to calculate the value of vanilla options we can start to examine the effect of stochastic volatility on today's and more importantly, future volatility surfaces:

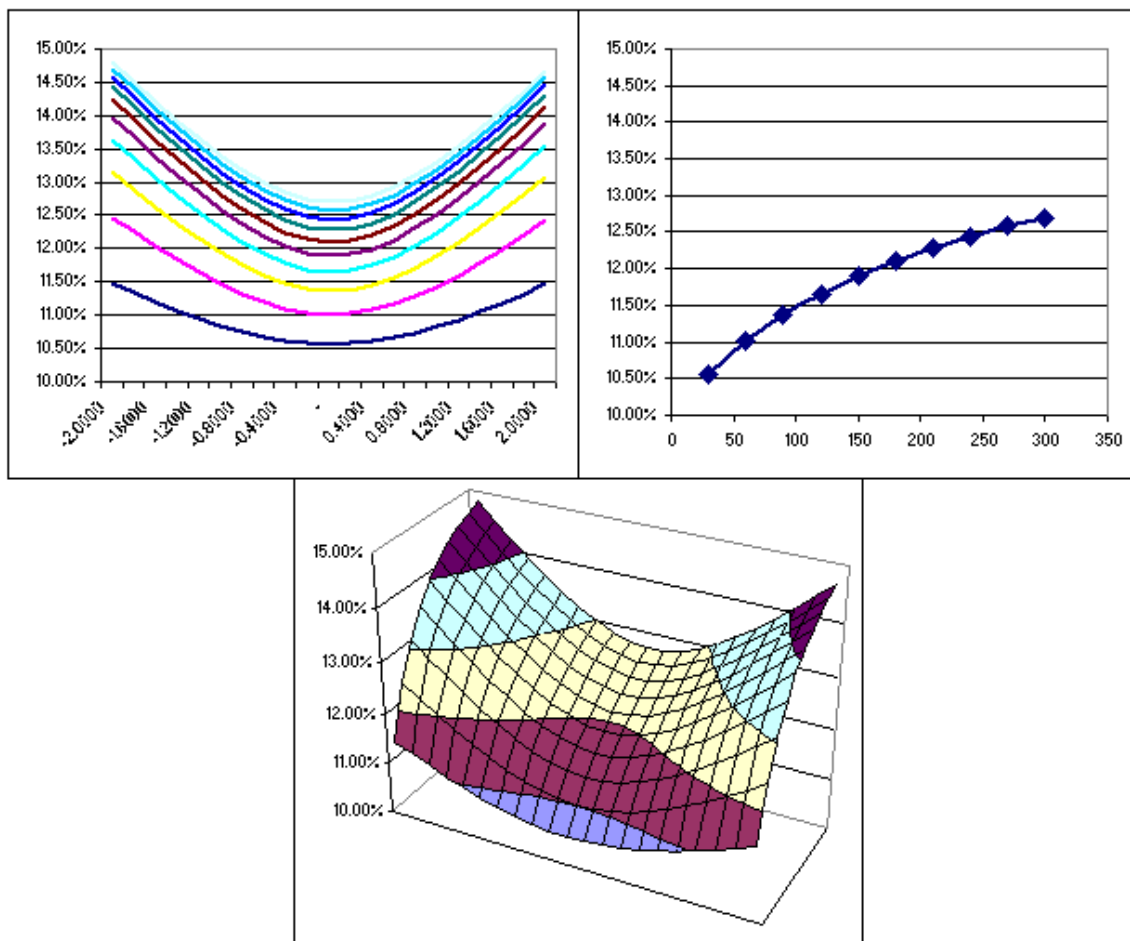
Example 1: $\alpha = 3$, $\sigma_0 = 10\%$, $\sigma_* = 15\%$, $\xi = 0\%$



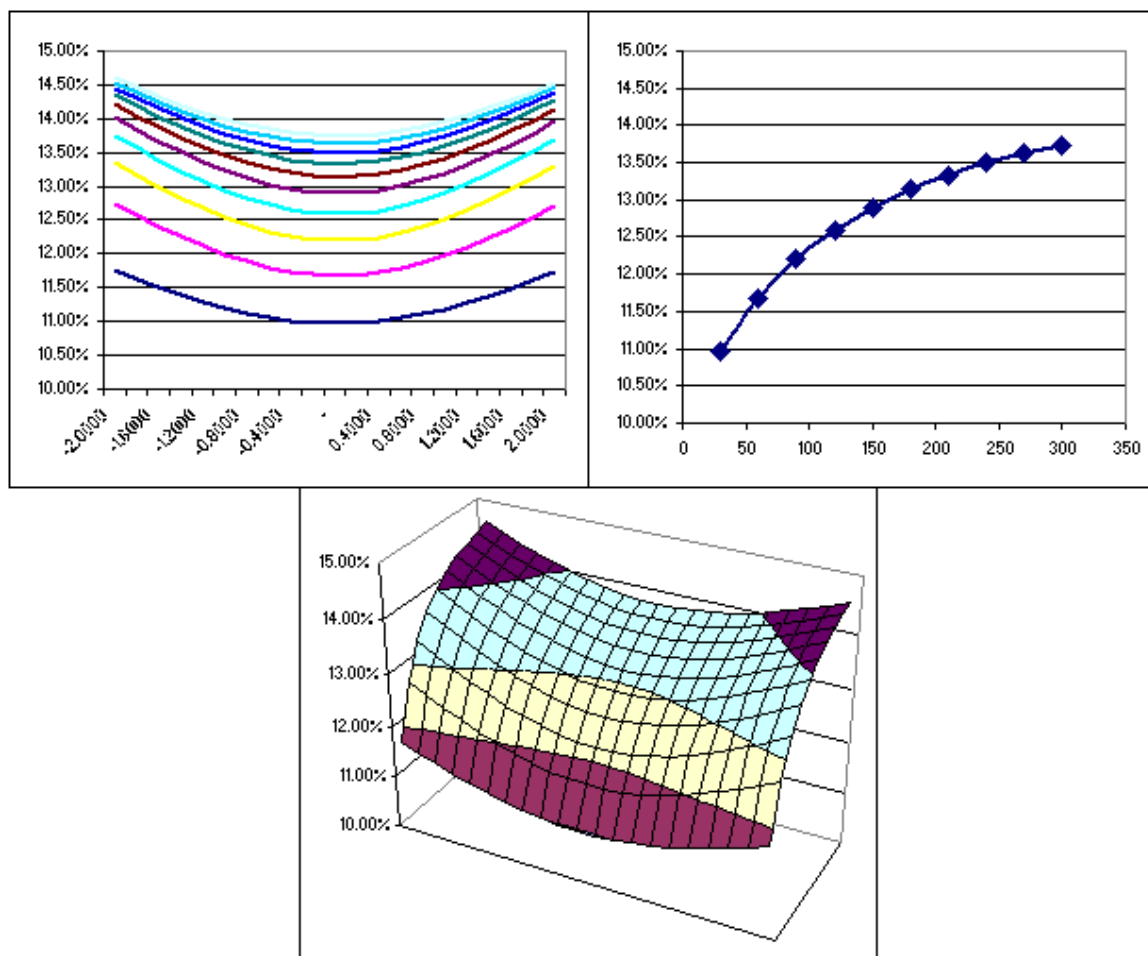
Example 2: $\alpha = 6$, $\sigma_0 = 10\%$, $\sigma_* = 15\%$, $\xi = 0\%$



Example 3: $\alpha = 3$, $\sigma_0 = 10\%$, $\sigma_* = 15\%$, $\xi = 300\%$



Example 4: $\alpha = 6$, $\sigma_0 = 10\%$, $\sigma_* = 15\%$, $\xi = 300\%$



Many models can calibrate to today's volatility surface - a stochastic volatility model can have the local volatility scalings applied to ensure this. However, whereas different models agree on today's volatility surface they will price exotic contracts differently as they each assume a different realisation of the forward dynamics of the volatility surface.

For the local-stochastic volatility model defined as

$$\begin{aligned} dS &= \mu S dt + \lambda(S, t) \sigma dW_1 \\ dX &= \alpha \left(\bar{X}^* - X \right) dt + \xi dW_2 \end{aligned}$$

Then when $\lambda(S, t) = 1$ the model is pure stochastic and when $\xi = 0$ the model is pure local.

At some future spot level $S(t)$ there will be a distribution for forward smiles where stochastic volatility exists, collapsing to a unique forward smile when the model reduces to a local volatility model.

We start, once again, with the pure stochastic volatility case. Recalling the calculation of the call price

$$\begin{aligned} C(T, K) &= \int_0^\infty C(V, T, K) \varphi(V) dV \\ &= \int_0^\infty \int_0^\infty (S - K) \psi(S|V) \varphi(V) dS dV \\ &= \int_0^\infty \int_0^\infty \int_0^\infty (S - K) \psi(S|V) \varphi(V|\sigma^2(T)) \vartheta(\sigma^2(T)) dS dV d\sigma^2(T) \end{aligned}$$

$\psi(S|V)$ is the distribution of spot conditional upon the total variance V and in this case is the standard lognormal density; $\varphi(V|\sigma^2(T))$ is the distribution of the total variance V conditional upon the instantaneous variance $\sigma^2(T)$ at time T ; and finally, $\vartheta(\sigma^2(T))$ is the distribution of the instantaneous variance

at time T . The density $\psi(S|V)$ is known and $\vartheta(\sigma^2(T))$ is obtained from

$$X(T) = X(0) + \bar{X}^* (1 - e^{-\alpha T}) + \frac{\xi}{\sqrt{2\alpha T}} (1 - e^{-2\alpha T})^{1/2} \bar{W}_T$$

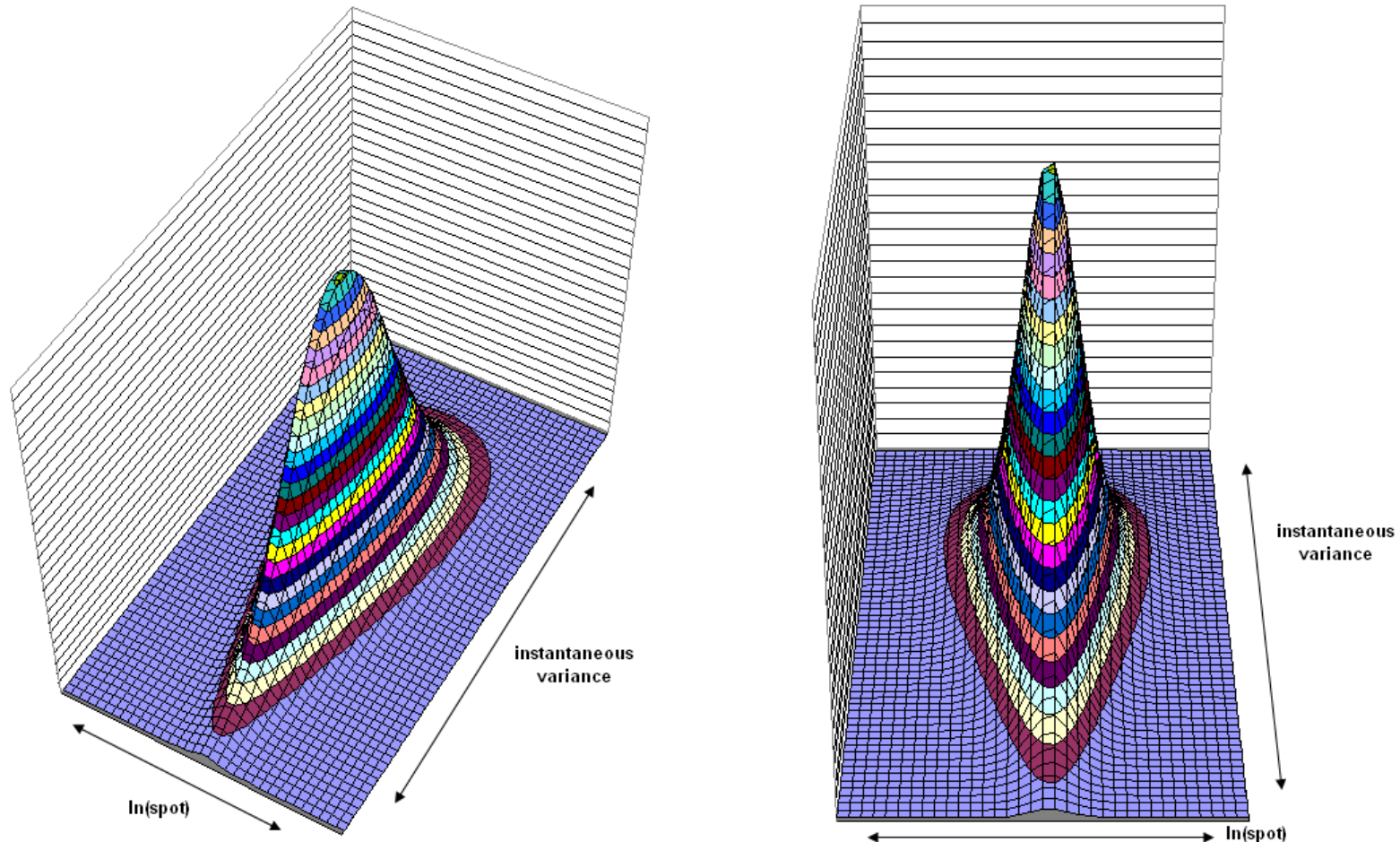
However, $\varphi(V|\sigma^2(T))$ needs to be calculated. Once again we can initially assume $\varphi(V|\sigma^2(T))$ is lognormal and calculate

$$E[V|\sigma^2(T)] = E\left[\frac{1}{T} \int_0^\infty \sigma^2(t) dt \middle| \sigma^2(T)\right] = \frac{1}{T} \int_0^\infty E[\sigma^2(t) | \sigma^2(T)] dt$$

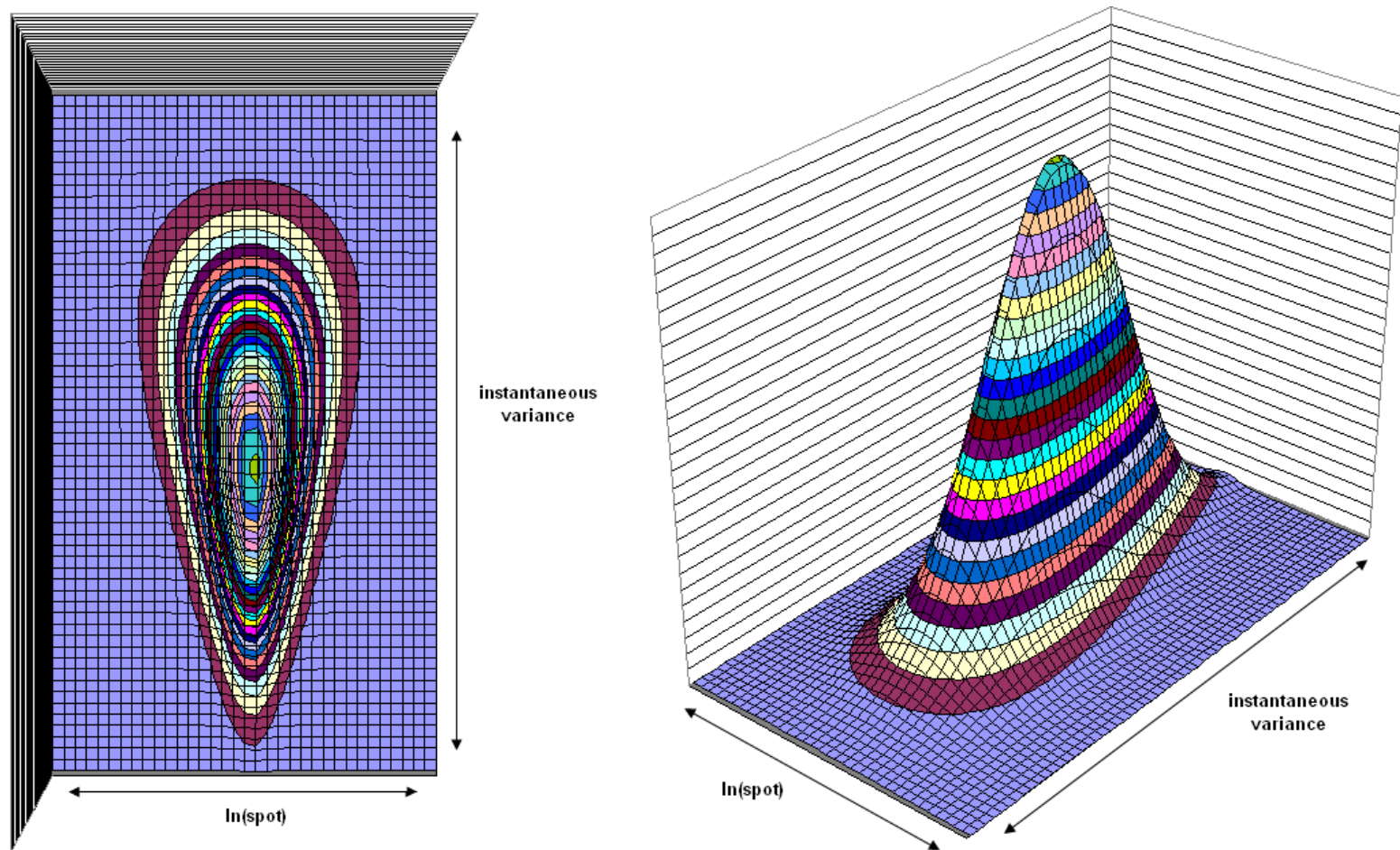
$$E[V^2|\sigma^2(T)] = E\left[\frac{1}{T^2} \int_0^\infty \int_0^\infty \sigma^2(t_1) \sigma^2(t_2) dt_1 dt_2 \middle| \sigma^2(T)\right] = \frac{1}{T^2} \int_0^\infty \int_0^\infty E[\sigma^2(t_1) \sigma^2(t_2) | \sigma^2(T)] dt_1 dt_2$$

Both $E[\sigma^2(t) | \sigma^2(T)]$ and $E[\sigma^2(t_1) \sigma^2(t_2) | \sigma^2(T)]$ can be calculated from Brownian bridge constructions.

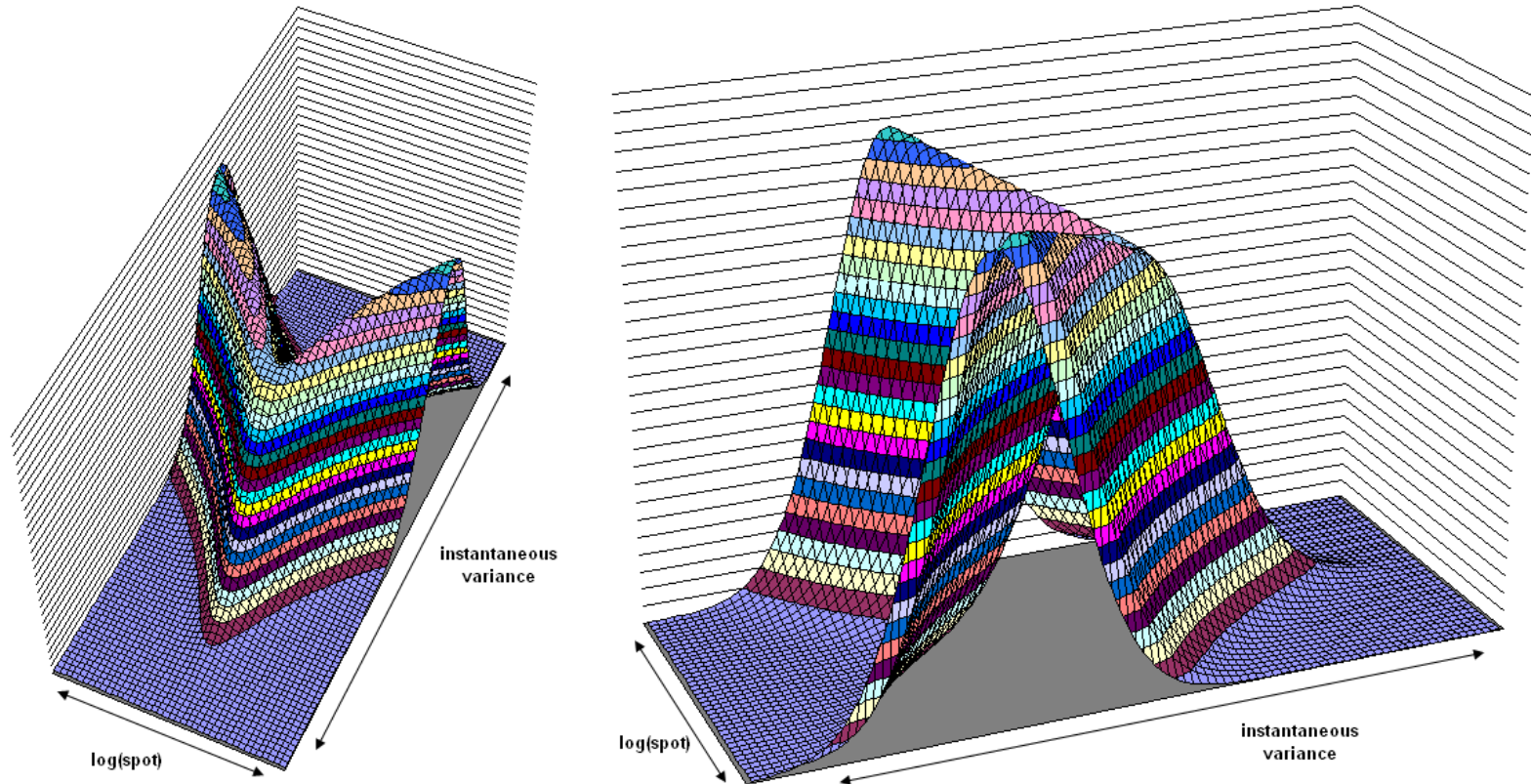
Here we show different views of the joint distribution of the $\ln(\text{spot})$ and the instantaneous variance at some future date.



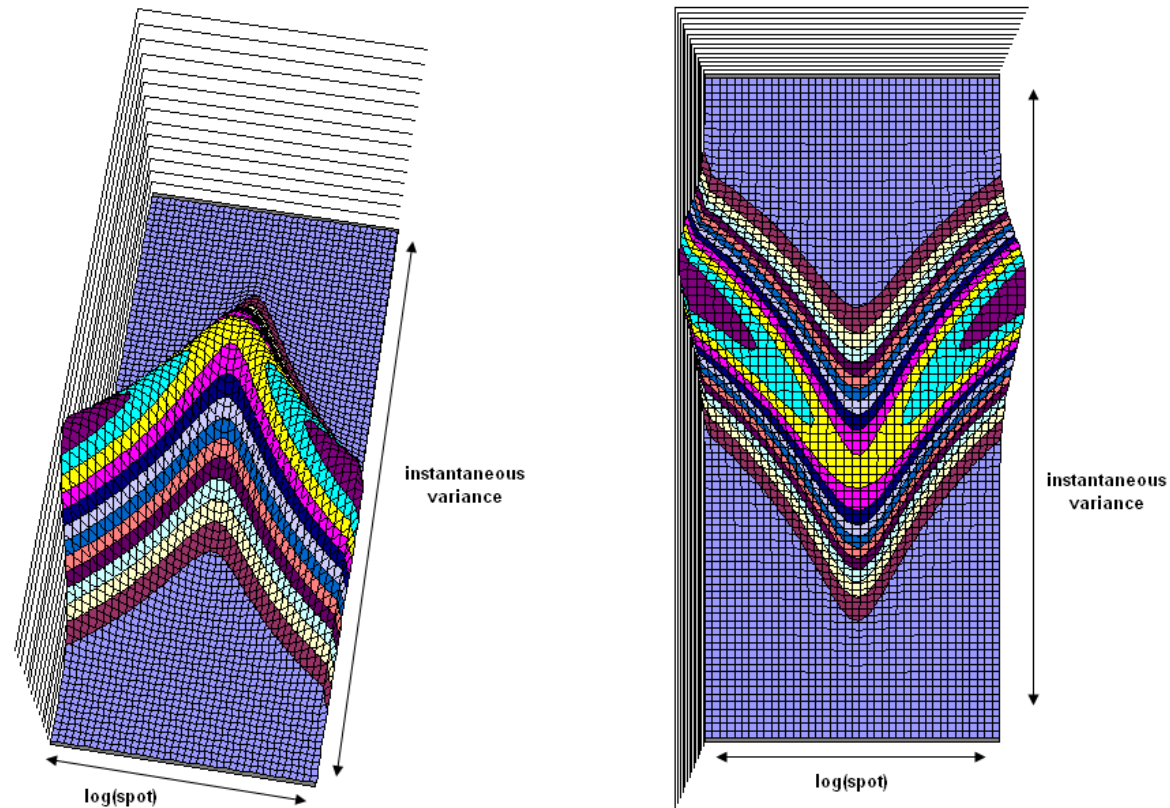
Here we show different view of the joint distribution of the $\ln(\text{spot})$ and the instantaneous variance at some future date.



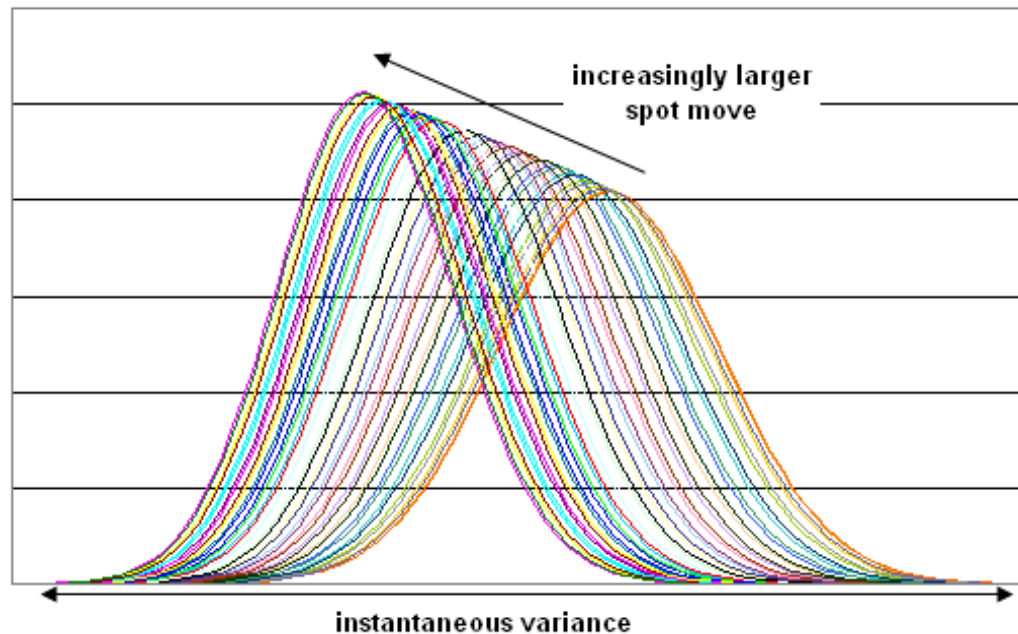
Here we show different views across $\log(\text{spot})$ the distribution of the instantaneous variance conditional upon spot at some future date.



Here we show different views across $\log(\text{spot})$ the distribution of the instantaneous variance conditional upon spot at some future date.



Of importance is given a spot level in the future, what is the distribution of instantaneous volatility. For each instantaneous volatility there corresponds a volatility surface. So we are in a position to understand the distribution of forward volatility surfaces for each future spot level.

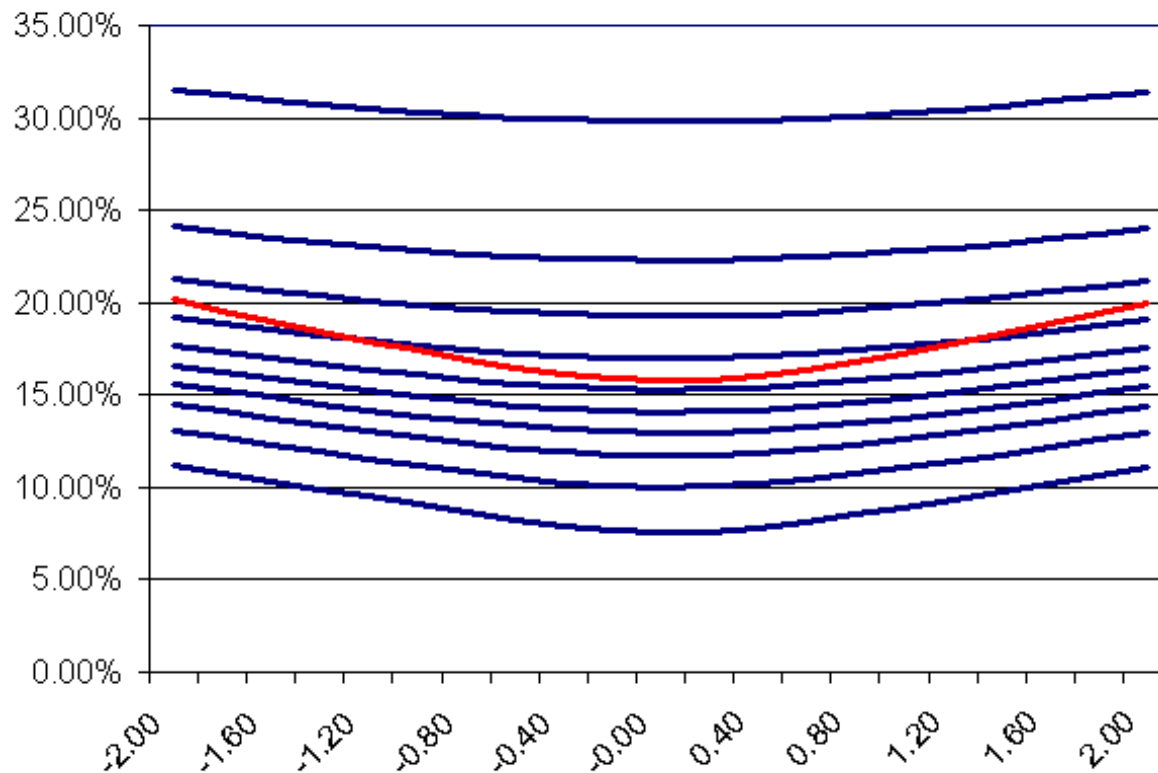


At a future spot level S and instantaneous volatility σ at time T if $\Pi(S, \sigma, T)$ is the associated implied volatility surface then we can define the implied volatility surface from the expected price at spot (S, T) as $\Theta(S, T) = \{\theta(S, T, K, T_{exp}) : 0 < K < \infty, T_{exp} > T\}$ where

$$C(S, T, K, T_{exp}, \theta(S, T, K, T_{exp})) = \int_0^\infty d\sigma \phi(\sigma|S) C(S, T, K, T_{exp}, \sigma_{imp}(\Pi(S, \sigma, T), K, T_{exp}))$$

Here $C(S, T, K, T_{\text{exp}}, \sigma_{\text{imp}}(\Pi(S, \sigma, T), K, T_{\text{exp}}))$ is the model value of the underlying contract at (S, σ, T) .

In the graph below for a future spot level the volatility distribution is subdivided into 10% sections and the implied volatility smile from the expected option prices are shown. The red line is the implied volatility smile for the entire distribution. The implied smile for the entire distribution has greater curvature (higher strangles) due to the presence of vol of vol.

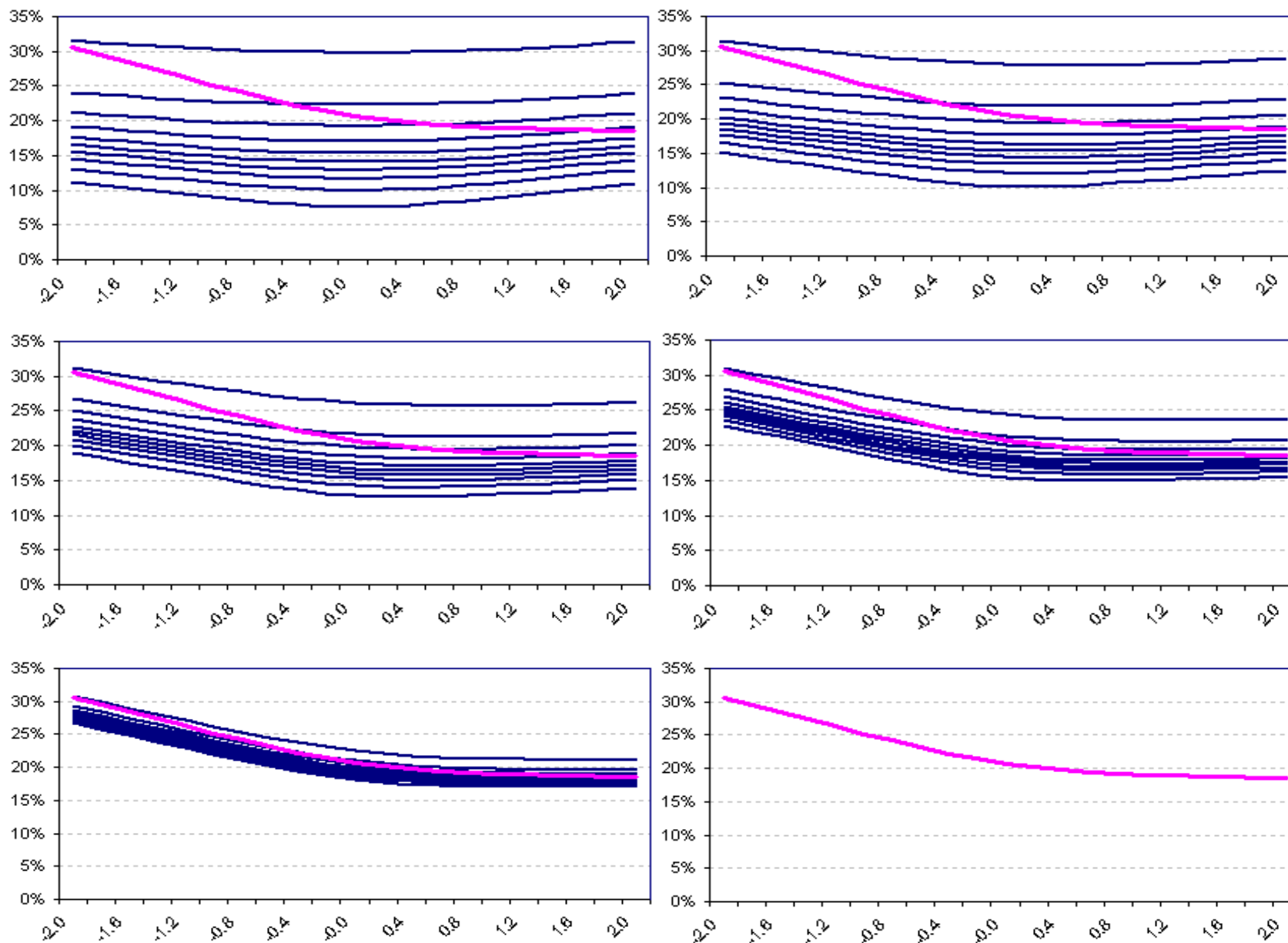


FX Quantitative Strategy Group

The Dynamics of the Forward Smile



We reduce the vol of vol on the LSVol model (blue lines) and see the forward smiles collapse onto the LVol (pink line) smile.



Hurdle Options

A hurdle option is defined as a contract which has a vanilla option payoff at time T_{exp} provided spot is not beyond the hurdle level H at the hurdle time T_{hurdle} . The hurdle level can be either upper or lower on an underlying call option. Consider here the case of a call option with strike K and a lower knockout hurdle level. Then the price of such an option can be represented

$$V_{\text{hurdle}} = DF \int_H^{\infty} dS (T_{\text{hurdle}}) C(S, T_{\text{hurdle}}, K, \theta(S, T, K, T_{\text{exp}}), T_{\text{exp}}) \varphi(S)$$

ie the integral across spot on the hurdle date of the value of the underlying option priced off the implied volatility surface $\Theta(S, t)$. So the value of a hurdle option is defined by today's volatility surface (which will give the spot distribution on the hurdle date) and the future volatility surface implied from the expected option prices.

Forward Volatility and Forward Smile Agreements

The forward volatility contract that trades within the FX market has a couple of variants dependent upon the currency pair in question:

- the agreement to enter into a delta-neutral straddle on the forward volatility agreement date
- the agreement to enter into a forward straddle on the forward volatility agreement date

The first case means that the strike for a given level of spot is a function of the ATM volatility on the FVA date and so is a function of the level of the instantaneous volatility. That requires special handling that will not be discussed here. For the second case then for the option part the value is given by

$$DF \int_0^{\infty} dSC(S, T_{FVA}, F(S(T_{FVA}), T_{\text{exp}}), \theta(S, T_{FVA}, F(S(T_{FVA}), T_{\text{exp}}), T_{\text{exp}}), T_{\text{exp}}) \varphi(S) \\ + DF \int_0^{\infty} dSP(S, T_{FVA}, F(S(T_{FVA}), T_{\text{exp}}), \theta(S, T, F(S(T_{FVA}), T_{\text{exp}}), T_{\text{exp}}), T_{\text{exp}}) \varphi(S)$$

which as in the hurdle case is a function only of $\Theta(S, t)$. More generally structures such as forward butterflies where the strike are defined relative to the future spot or forward rate (ie not in terms of delta which is volatility level dependent) can be calculated as above.

Note: An alternative definition of the forward smile

For the bulk of pricing problems then the forward smile defined previously - as the implied volatility surface from the future expected option prices - is appropriate. There is an alternative forward smile defined as $\bar{\Theta}(S, T) = \{\bar{\theta}(S, T, K, T_{exp}) : 0 < K < \infty, T_{exp} > T\}$ where

$$\bar{\theta}(S, T, K, T_{exp}) = \int_0^{\infty} d\sigma \sigma_{imp}(\Pi(S, \sigma, T), K, T_{exp})$$

This is the direct expectation of the implied volatilities themselves across the distribution of volatility surfaces associated with each instantaneous volatility level. This is of use in the small category of products which have payoff a function of the implied volatility itself.

Concluding remarks

In this presentation we have reviewed local volatility, stochastic volatility and their combination in the local-stochastic volatility model framework. We have looked at numerical schemes - Crank-Nicholson and ADI - that allow the reader to implement these models. Further we have begun to investigate the forward smile within these models with a view to understanding intuitively how the models construct prices and allow us to consider building a model by construction of the forward smiles themselves.

Further topics for the Advanced Modelling Volatility course (!):

- the role of correlation on the forward smile and so on pricing
- the dynamics of the market smiles and their understanding via PCA / ICA techniques
- the effect of stochastic interest rates and the forward smile under more general models
- path dependency - construction of the appropriate forward smile for exotics and the relationship to how traders make prices.

Selected further reading

Please see the below and references therein for some background information to this presentation:

- H. Cramér (1957). *Mathematical Methods of Statistics*. Princeton University Press, Princeton.
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- M. Overhaus, A. Ferraris, T. Knudsen, R. Milward, L. Nguyen-Ngoc, G. Schindlmayr. *Equity Derivatives: Theory and Applications*. Wiley Finance, 2002.

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