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Publication date:
2008

Document version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Hansen, B. O., & Keiding, H. (2008). *How Likely Is Factor Price Equalization?* Department of Economics, University of Copenhagen.

Discussion Papers
Department of Economics
University of Copenhagen

No. 08-10

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ISSN: 1601-2461 (online)

How Likely Is Factor Price Equalization?

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February 2008

Abstract

The idea of treating factor price equalization as a situation, where the distribution of factors among countries is compatible with an equilibrium in an integrated world economy, has been refined to give the so-called lens condition for factor price equalization. In this paper, we show that the lens condition may be used to give estimates for the probability of factor price equalization when factors are distributed randomly among countries and, in addition, the techologies are sampled according to a given probability distribution. The estimates indicate that factor price equalization may occur less often than intuitively conceived.

Keywords: international trade, factor price equalization, lens condition

JEL classification: F11

1. Introduction

The factor price equalization (FPE) theorem, found by Lerner (1952) and Samuelson (1948, 1949, 1953) is one of the most remarkable achievement of international trade theory. It says that under suitable – and, as it turns out, rather restrictive – assumptions, factors of production will obtain the same remuneration in countries trading only in final products. Subsequent authors have refined and reformulated it in several ways, one of the most fruitful being the introduction by Dixit and Norman (1980) of the factor price equalization domain, the set of initial distributions of factors among countries such that international trade equilibria are identical to equilibria of an integrated world economy with no restriction on trade in factors.

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Following a geometric approach to FPE domains, Deardorff (1994) formulated the so-called lens condition for FPE, which points to the role of the factor proportions in the different sectors which span the FPE domain. The lens condition has been intensively discussed in the literature, see e.g. Deardorff (2001), Demiroglu and Yun (1999), Kemp (2001), Kemp and Okawa (1998), Qi (2003), Wong and Yun (2003), Yun (2003), Chakrabarti (2006), adapting it to various purposes. One of the main advantages of this approach is that it lends itself easily to geometric reasoning. Intuitively one has that if the vectors of inputs needed in the different industries in the integrated equilibrium have very different directions, then the FPE domain will be large relative to the set of all possible distributions among countries of factor endowments, and if for given world endowment, each distribution of endowment among countries is considered equally likely, then the probability of FPE is large.

However, restricting attention to distribution of given endowments, taking world factor endowment and technology as given, appears as a not very useful approach, and consequently we shall allow also the technologies for producing the commodities to vary. More specifically, we assume that endowments as well as technologies are chosen at random, and we are then interested in the probability of factor price equalization, both for fixed technology and for the general case. In the textbook case, the resulting assessment of the probability of factor price equalization is more or less as would be expected, showing it to occur in close to half of the possible cases; the fact that it is actually less than one half is due to the method of sampling technologies and may not cause too much concern. When moving to more than two commodities and factors, things change rapidly, in the sense that the probability of factor price equalization decreases towards zero with rising dimension.

This somewhat unexpected result, that factor price equalization becomes increasingly unlikely as the number of goods and factors increases, is of course in its turn dependent on the particular way of parametrizing technologies as well as the choice of probability distribution on parameters. On the other hands, since the assessments are bases on considerations of volumes of suitable convex sets, that is on Lebesgue measure, they seem to catch some basic features of the problem, which tend to be overlooked when considering only low-dimensional cases.

The paper is organized as follows: In Section 2, we introduce notation and present a version of the condition for factor price equalization which largely follows the literature. Then we turn to our main notion, the probability of factor price equalization, which is considered in the classical case with equal number of commodities and factors in Section 3, and in Section 4 in the general case. We conclude in Section 5 with some general comments. A technical result needed in Section 3 is proved in an appendix.

2. Definitions. The lens condition

In this section, we introduce the main concepts and the notation to be used in the paper. Also we state a version of the well-known lens condition for factor price equalization which is adapted to our purpose.

We start by introducing technologies, using support functions of the 1-isoquant. More specifically, let $\triangle_r = \{q \in \mathbb{R}^r_+ \mid \sum_{h=1}^r q_h = 1\}$ be the set of normalized prices of r given factors of production, and let $\mathcal{S}_r$ be the set of concave functions $\sigma : \triangle_r \to \mathbb{R}_+$. We parametrize the family of admissible production functions by the elements of $\mathcal{S}_r$, interpreted as (downwards) support functions of 1-isoquants, so that the upper level set at output 1 (all factor combinations yielding at least output 1) is

$$Y_\sigma = \{z \in \mathbb{R}^r_+ \mid \forall q \in \triangle_r,\, q \cdot z \geq \sigma(q)\},$$

and the production function associated with σ is f_σ given by

$$f_\sigma(z) = \max\{\lambda \mid \lambda^{-1} z \in Y_\sigma\}.$$

Since elements of $\mathcal{S}_r$ are (support functions of) 1-isoquants, we refer to them as techniques rather than as production functions.

Let $\phi_\sigma(q) = \{z \in \mathbb{R} \mid q \cdot z = \sigma(q)\}$ be the set of cost minimizing vectors of factor input at the price q. Clearly, for each $q \in \triangle_r$, the set $\phi_\sigma(q)$ is closed and convex, and $\phi_\sigma(q)$ is a singleton if Y_σ is strictly convex.

In the following, a *technology* is an array $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n) \in (\mathcal{S}_r)^n$ consisting of n elements of $\mathcal{S}_r$. In the interpretation, $\underline{\sigma}$ specifies the method of producing n distinct commodities using the r factors of production. The following standard property of technologies is a useful consequence of the concavity of the elements of $\mathcal{S}_r$.

LEMMA 1. *Let $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n) \in (\mathcal{S}_r)^n$ and $\omega \in \mathbb{R}^r_{++}$ be given, and let $y^0 = (y^0_1, \ldots, y^0_n) \in \mathbb{R}^n_{++}$. Then the following are equivalent:*

(i) y^0 maximizes some quasiconcave and monotonic function $U(y)$ over all $y = (y_1, \ldots, y_n)$ for which there are $z_1, \ldots, z_n \in \mathbb{R}^r_+$ with $y_i = f_{\sigma_i}(z_i)$ and $\sum_{i=1}^n z_i = \omega$,

(ii) there is $q \in \triangle_r$ such that

$$\omega \in \sum_{i=1}^n y^0_i \phi_{\sigma_i}(q). \tag{1}$$

PROOF: (i)⇒(ii): Let

$$X^0 = \{z \in \mathbb{R}^r_+ \mid \exists (z_1, \ldots, z_n),\, U(f_{\sigma_1}(z_1), \ldots, f_{\sigma_n}(z_n)) \geq U(y^0_1, \ldots, y^0_n)\}.$$

By convexity of production sets and quasi-concavity of U, this set is convex. By our assumptions, it intersects the set $\{z \in \mathbb{R}^r_+ \mid z \leq \omega\}$ only in ω. By separation of convex sets, there is $q \in \triangle_r$ such that $q \cdot x \geq q \cdot \omega$ for all $x \in X^0$. Using the definition of X^0 we see that for each i, $q \cdot z_i \geq q \cdot z_i^0$ for all z_i such that $f_{\sigma_i}(z_i) \geq f_{\sigma_i}(z_i^0)$, where $f_{\sigma_i}(z_i^0) = y_i^0$, each i, and $\sum_{i=1}^n z_i^0 = \omega$.

(ii)⇒(i): Write $\omega = z_1 + \cdots + z_n$ with $z_i \in y_i^0 \phi_{\sigma_i}(q)$ for each i, and define $p_i = (q \cdot z_i)/y_i^0$. Then by constant returns to scale we have that $p_i f_{\sigma_i}(z_i') \leq q \cdot z_i'$ for all $z_i' \in \mathbb{R}_+^r$, and it follows that y^0 maximizes $p \cdot \sum_{i=1}^n f_{\sigma_i}(z_i')$ over all $(z_1', \ldots, z_n') \in (\mathbb{R}_+^r)^n y$ with $\sum_i z_i' = \omega$, which is (i). □

The situation considered in Lemma 1 corresponds to what is called "the integrated equilibrium" in the literature on factor price equalization (we have considered Pareto efficient allocations rather than equilibria in an integrated economy, which under our assumptions amounts to the same). We now proceed to consider the case where total factor endowments ω are distributed among the K countries according to the array $\underline{\omega} = (\omega^1, \ldots, \omega^K) \in \left(\mathbb{R}_+^r\right)^K$. The array $(\underline{\sigma}, \underline{\omega})$ is called a K-country world; the associated 1-country world $(\underline{\sigma}, \omega)$, where $\omega = \sum_{k=1}^K \omega^k$, is called the *integrated economy.*

An *equilibrium* in the K-country world $(\underline{\sigma}, \underline{\omega})$ is an array $(p, (y^k, q^k)_{k=1}^K) \in \triangle_n \times (\triangle_r \times \mathbb{R}_+^n)^K$ such that for each k,

$$y^k \text{ maximizes } p \cdot y \text{ over all } y \text{ such that } \omega_k \in \sum_{i=1}^n y_i \phi_{\sigma_i}(q^k). \tag{2}$$

The equilibrium is a *factor price equalization equilibrium* (FPEE) if $q^k = q^l$ for $k, l = 1, \ldots, K$.

In the notation introduced thus far, the "lens condition" for factor price equalization takes the following form:

PROPOSITION 1. *Let $(\underline{\sigma}, \underline{\omega})$ be a K-country world, and let $(p, (y^k, q^k)_{k=1}^K)$ be an equilibrium with $y^0 = \sum_{k=1}^K y^k$. Then the following are equivalent:*

(i) $(p, (y^k, q^k)_{k=1}^K)$ is an FPEE with $q^1 = \cdots = q^K = q$,

(ii) there is $q \in \triangle_r$ such that

$$\omega^k \in \sum_{i=1}^n \text{conv}\left(\{0\}, y_i^0 \phi_{\sigma_i}(q)\right), \tag{3}$$

for all k.

PROOF: (i)⇒(ii): Since (2) holds with $q^k = q$ for each k, we have that $\omega^k \in \sum_{i=1}^n y_i^k \phi_{\sigma_i}(q)$ for each k, and since $y_i^k \phi_{\sigma_i}(q) \subset \text{conv}\left(\{0\}, y_i^0 \phi_{\sigma_i}(q)\right)$ for each i and k, we have (3).

(ii)⇒(i): By (3), the factor endowment ω_k has a representation

$$\omega_k = z_1^k + \cdots + z_n^k,$$

where $z_i^k \in y_i^k \phi_{\sigma_i}(q))$ for some $y_i^k \in [0, y_i^0]$, each k, and with $y_i^0 = \sum_{k=1}^K y_i^k$ for each i. We then have that

$$\omega = \sum_{k=1}^{K} \omega^k \in \sum_{k=1}^{K} y_i \phi_{\sigma_i}(q),$$

which by Lemma 1 means that $(p, (y^0, q))$ is an equilibrium in the integrated economy for some $p \in \triangle_n$. It now follows immediately that $(p, (y^k, q)_{k=1}^K)$ is an FPEE. $\square$

It should be noticed that the present characterization of FPEEs allows for technologies for which isoquants may not be smooth. A prominent such case is that of Leontief technologies: A technique $\sigma \in \mathcal{S}_r$ is Leontief if it has the form σ_a with

$$\sigma_a(q) = q \cdot a$$

for some $a \in \mathbb{R}_+^r$. We denote by $\mathcal{S}_r^L$ the set of Leontief techniques. A technology $\underline{\sigma}$ is Leontief if it belongs to $\left(\mathcal{S}_r^L\right)^n$, that is if $\sigma_i = \sigma_{a_i}$ for each i, and the output y^0 maximizes $p \cdot y$ for some $p \in \triangle_n$ using total resources ω if

$$\sum_{i=1}^{n} y_i^0 a_i \leq \omega$$

with equality for at least one of the r coordinates. We have that

$$\sum_{i=1}^{n} y_i^0 a_i = \sum_{i=1}^{n} (p_i y_i^0) \left[\frac{1}{p_i} a_i\right] = p \cdot y^0 \left(\sum_{i=1}^{n} \frac{p_i y_i^0}{p \cdot y^0} \left[\frac{1}{p_i} a_i\right]\right),$$

so that the solution is identical to that of finding maximal output in a one-good economy with the composite good obtained by valuing n-tuples at prices p and using the technique σ defined by

$$\sigma(q) = \min\left\{\frac{1}{p_1}\sigma_{a_1}(q), \ldots, \frac{1}{p_n}\sigma_{a_n}(q)\right\},$$

and in the solution, q supports each of the points $\frac{1}{p_i} a_i$, $i = 1, \ldots, n$.

A similar result holds in the general case, and it provides the link between the classical approach to FPE, finding the (unique) factor prices corresponding to commodity prices p. Except for uniqueness, this carries over to our present setup.

PROPOSITION 2. *Let $(\underline{\sigma}, \underline{\omega})$ be a K-country world, and let $(p, (y^k, q^k)_{k=1}^K)$ be an equilibrium with $y^0 = \sum_{k=1}^K y^k$. Then the following are equivalent:*

(i) $(p, (y^k, q^k)_{k=1}^K)$ *is an FPEE with* $q^1 = \cdots = q^K = q$,

(ii) there is $q \in \triangle_r$ *such that* $\sigma_i(q) = p_i$ *for each* i *and*

$$\omega^k \in (p \cdot y^k)\,\mathrm{conv}\left(\left\{\frac{1}{p_i}\phi_{\sigma_i}(q) \,\middle|\, i = 1, \ldots, n\right\}\right) \tag{4}$$

for each k.

PROOF: (i)⇒(ii): Using property (2) of the equilibrium we get that

$$\omega^k \in \sum_{i=1}^n p_i y_i^k \frac{1}{p_i}\phi_{\sigma_i}(q) = (p \cdot y^k) \sum_{i=1}^n \lambda_i \frac{1}{p_i}\phi_{\sigma_i}(q)$$

with $\lambda_i = p_i y_i^k/(p \cdot y^k)$ for each i, so that (4) holds for each k.

(ii)⇒(i): Suppose that (4) holds for each k. Adding over k we get that

$$\omega = (p \cdot y^0) \sum_{i=1}^n \lambda_i \frac{1}{p_i} z_i = \sum_{i=1}^n \frac{\lambda_i (p \cdot y^0)}{p_i} z_i$$

with $z_i \in \phi_i(q)$, each i. Using that $q \cdot z_i' = \sigma_i(q) = p_i$, all $z_i' \in \phi_{\sigma_i}(q)$, each i we have that $(q \cdot z_i')/p_i$ is independent of i, so that q must be a support of $\{z_i \mid f_{\sigma_i}(z_i) \geq 1\}$ for each i, and consequently we may assume that $q^1 = \cdots = q^K = q$, which is (i). □

3. Assessing the probability of FPE: The classical case ($r = n$)

In this section, we consider the probabilistic approach to factor price equalization in its classical version where the number of traded commodities equals the number of factors. We start with the case of only two factors of production, allowing for a graphical representation using the Edgeworth box. This representation also lends some intuition to the probability of factor price equalization, in least in the simple case of independent and uniform distribution of resources, as the relative area of the subset of the box bounded by the rays of factor inputs in the integrated equilibrium.

For simplicity let the technology be Leontief, with $(\sigma_1, \sigma_2) = (\sigma_{a_1}, \sigma_{a_2})$. If the technology is fixed, and the commodity prices are given, then the factor equalization domain can be represented in the Edgeworth box as the area of the set A between the rays from the origin through a_1 and a_2 (in the coordinate systems with origin O_1 as well as in that with origin O_2), cf. Fig.1.

We assume that factor endowments in each country are drawn independently according to a probability distribution F with density f. If $\Pi_{\underline{\sigma},p}$ is the probability of factor price equalization, then

$$\Pi_{\underline{\sigma},p} = \int_{(z_1,z_2)\in A} f(z) f(\omega - z) dz. \tag{5}$$

In order to obtain explicit assessments of $\Pi_{\underline{\sigma},p}$ we assume that F is the uniform distribution on $[0,1]^2$, and that $\omega = (1,1)$, so that the expression in (5) equals the area of A. Using the notation as indicated in Fig.1, we find that the line segment BC has length $2\alpha\beta/(\alpha+\beta)$, so that the probability of FPE is

$$\Pi_{\underline{\sigma},p} = 2\sqrt{2}\frac{\alpha\beta}{\alpha+\beta} \tag{6}$$

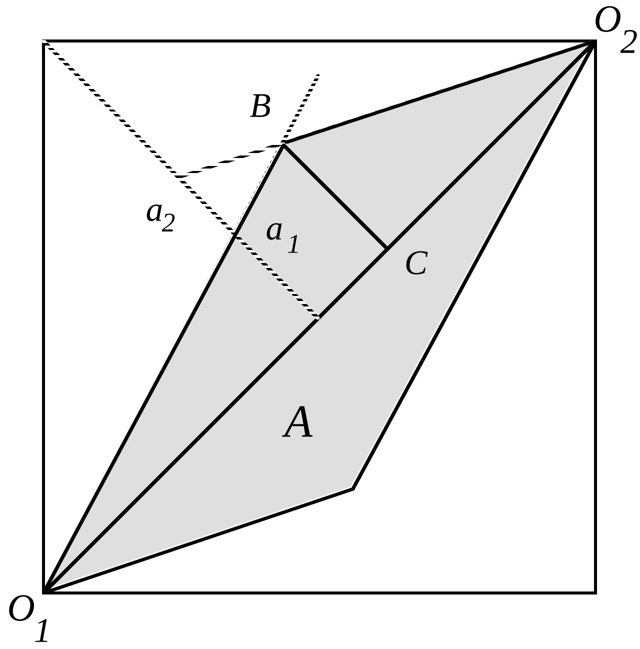

Figure 1

in the case illustrated in Fig.1.

It is evident that the quantity in (6) depends on the technology as expressed here by the two rays $\phi_{\sigma_1}(q), \phi_{\sigma_2}(q)$. Following the probabilistic approach, we would proceed to take expectations over technologies $\underline{\sigma}$ and commodity prices p. This however presupposes a given probability distribution over the set $\left(\mathcal{S}_2^L\right)^2$; such a distribution with reasonable intuitive content does not suggest itself except in the particular case, where all techniques in $\mathcal{S}_2$ are Leontief, and therefore we restrict our treatment to this case.

Thus, we assume that Leontief technologies are parametrized by pairs (a_1, a_2) of elements of the simplex $\{a \in \mathbb{R}^2_+ \mid a_1 + a_2 = 1\}$ which define the relevant factor proportions in each of the two techniques. We assume that the two techniques are sampled in such a way that total factor endowment can be exploited efficiently, meaning that $a_{11} \geq \frac{1}{2}, a_{22} \geq \frac{1}{2}$. The case where one of these inequalities is violated will be considered separately.

In principle, we need not only factor proportions but also the amount of factors needed to produce one unit of commodity. However, the factor price equalization domain will be determined only by factor proportions; the absolute factor productivity will matter only for the commodity prices, and with our parametrization of technologies, we need not take the latter into account. We get the following result in the $2 \times 2 \times 2$ case.

PROPOSITION 3. *Consider the family* $\left(\mathcal{S}_2^L\right)^2 \times \{(\omega_1, \omega_2) \in (\mathbb{R}^2_+)^2 \mid \omega_1 + \omega_2 = (1,1)\}$ *of 2-country worlds with 2 commodities and 2 factors of production, where techniques and endowments are sampled uniformly given that factors can be used efficiently. Then the* *probability of factor price equalization* $\Pi^{2,2,2}$ *satisfies*

$$\Pi^{2,2,2} < \frac{1}{2}.$$

PROOF: Using the parametrization $u = \|a_1 - \left(\frac{1}{2}, \frac{1}{2}\right)\|$, $\beta = \|a_2 - \left(\frac{1}{2}, \frac{1}{2}\right)\|$, we have the following expression

$$\Pi^{2,2,2} = 2\sqrt{2} \int_0^{\frac{\sqrt{2}}{2}} \int_0^{\frac{\sqrt{2}}{2}} \frac{\alpha\beta}{\alpha+\beta}\, d\alpha\, d\beta.$$

After substitution of y for $\alpha + \beta$ we get that

$$\begin{aligned}
\Pi^{2,2,2} &= 2\sqrt{2} \int_0^{\frac{\sqrt{2}}{2}} \int_\beta^{\beta+\frac{\sqrt{2}}{2}} \left(1 - \frac{\beta}{y}\right) dy\, d\beta \\
&= 2\sqrt{2} \int_0^{\frac{\sqrt{2}}{2}} \left[\frac{\sqrt{2}}{2}\beta - \beta^2 \ln\left(1 + \frac{\sqrt{2}}{2}\frac{1}{\beta}\right)\right] d\beta
\end{aligned}$$

and using Taylor expansion of $\ln\left(1 + \frac{\sqrt{2}}{2}\frac{1}{\beta}\right)$ around $\ln 1 = 0$ we get that

$$\ln\left(1 + \frac{\sqrt{2}}{2}\frac{1}{\beta}\right) > \frac{\sqrt{2}}{2}\frac{1}{\beta} - \frac{1}{4\beta^2},$$

so that

$$\Pi^{2,2,2} > 2\sqrt{2} \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{4}\, d\beta = \frac{1}{2},$$

which gives the assessment of the proposition. □

The fact that factor price equalization occurs with probability less than $\frac{1}{2}$ may come as a surprise, since the "average" technology would be that where $\alpha = \beta = \frac{\sqrt{2}}{4}$, for which the area of the factor price equalization domain is exactly $\frac{1}{2}$. Needless to say, the result depends on the distribution of techniques; if we had chosen a uniform distribution over the angles between the diagonal and the factor proportion rays, the cases of instances of small factor price equalization domains would have weighted less and final probability would have been greater. However, the approach taken seems more natural from an economic point of view, using factor bundles rather than factor proportions, and it is much more easy to generalize to more than 2 factors of production.

For the extension of the results to cases of more than two commodities and factors, we start by considering the case $n = r = 3$ (and, as previously, $K = 2$). With total factor endowment $\omega = e = (1, 1, 1)$, we have that a 2-country world is defined by specifying three Leontief techniques σ_{a_1}, σ_{a_2}, σ_{a_3} with $a_1, a_2, a_3 \in \triangle_3$. By Proposition 2, factor price equalization will obtain whenever (ω_1, ω_2) is such that $\omega_i \in$ cone $(\{a_1, a_2, a_3\})$ for $i = 1, 2$. Geometrically, this means that ω_1 should belong to the set

$$\text{cone}\,(\{a_1, a_2, a_3\}) \cap [\{e\} - \text{cone}\,(\{a_1, a_2, a_3\})]. \tag{7}$$

From this we obtain a bound for the probability of factor price equalization, $\Pi^{3,3,2}$, namely

$$\Pi^{3,3,2} \leq 2\text{Vol}\left(\text{cone}\,(\{a_1, a_2, a_3\}) \cap \left\{z \in \mathbb{R}^3_+ \mid \Sigma_i z_i = 1\right\}\right), \tag{8}$$

and this expression may be used to obtain a numerical assessment of the bound. We need a further notion: Let P_3 be the expected value of the area of a triangle spanned by three points in the simplex $\triangle_3$ chosen at random, measured relative to the area of $\triangle_3$.

PROPOSITION 4. *Consider the family* $\left(\mathcal{S}_3^L\right)^3 \times \{(\omega_i)_{i=1}^3 \in (\mathbb{R}^3_+)^2 \mid \sum_i \omega_i = (1,1,1)\}$ *of 2-country worlds with 3 commodities and 3 factors of production, where techniques and endowments are sampled uniformly given that factors can be used efficiently. Then the probability of factor price equalization* $\Pi^{3,3,2}$ *satisfies*

$$\Pi^{3,3,2} < \frac{\sqrt{3}}{2} P_3 < \frac{\sqrt{3}}{2}\left(\frac{1}{2}\right)^2. \tag{9}$$

PROOF: Using (8), we have that the relative area of the factor price equalization at any choice of Leontief technology $(\sigma_{a_1}, \sigma_{a_2}, \sigma_{a_3})$ must be bounded from above by twice the relative area of

$$\text{conv}(\{0, \frac{\sqrt{3}}{2}a_1, \frac{\sqrt{3}}{2}a_2, \frac{\sqrt{3}}{2}a_3\}) = \frac{1}{2}\frac{\sqrt{3}}{2}\frac{m(\text{conv}(\{a_1, a_2, a_3\}))}{m(\triangle_2)},$$

where $m(\cdot)$ denotes area (or Lebesgue measure). Taking expectations over a_1, a_2, a_3 we get that

$$\Pi^{3,3,2} \leq 2\frac{1}{2}\frac{\sqrt{3}}{2}P_3,$$

and inserting the value of P_3 from Lemma 2 (in the appendix), we get (9). □

Comparing the expression in (9) to the result obtained in Proposition 3, we notice that the bound obtained is not exact, being based on the two cones spanned by the techniques from each of the end points of the (three-dimensional) Edgeworth box, which only in exceptional cases is identical to the factor price equalization domain. Even so, it is seen that the probability of factor price equalization is smaller in dimension 3 than in dimension 2. The assessment in dimension 3 can be generalized to higher dimensions with the same line of proof, which is left to the reader. The key ingredient here as above is assessment of expected volume of a subsimplex of $\triangle_n$ obtained by random selection of its verices, the quantity P_n considered in the appendix.

PROPOSITION 5. *Consider the family* $\left(\mathcal{S}_r^L\right)^r \times \{(\omega_i)_{i=1}^r \in (\mathbb{R}_+^r)^2 \mid \sum_i \omega_i = e\}$ *of 2-country worlds with* r *commodities and* r *factors of production, where techniques and endowments are sampled uniformly given that factors can be used efficiently. Then the probability of factor price equalization* $\Pi^{r,r,2}$ *satisfies*

$$\Pi^{r,r,2} < \frac{\sqrt{r}}{2} P_r < \frac{\sqrt{r}}{2} \left(\frac{1}{2}\right)^{r-1}. \tag{10}$$

Since the bound in (10) has the magnitude of $\frac{\sqrt{r}}{2^r}$, it goes to zero for $r \to \infty$. With a view to the large number of distinct commodities figuring in international trade as well as the large number of factors of production used in real life, it seems that factor price equalization is a rather unlikely event. Clearly, one should not overdo the importance of results as those obtained here, which pertain to a model of international trade which is anyway lacking in realism. But the result does point to a weakness of the classical theory which may have given too much attention to a phenomenon turning out to be specific for the low-dimensional geometric versions of the model. We return to this point in the concluding remarks.

4. Assessing the probability of FPE: Unequal numbers of commodities and factors

In the present section, we move beyond the classical case of equal number of commodities and factors of production, so that we have either (i) $r > n$ or (ii) $r < n$. Case (i) is rather easily resolved: If q is a common factor price vector, then the factor proportion vectors $\phi_{\sigma_1}(q), \ldots, \phi_{\sigma_n}(q)$ span a subspace of $\mathbb{R}_+^r$ of less than full dimension, so that the relative volume of the factor price equalization domain is 0 for all choices of technology. Thus, more factors than commodities means that factor price equalization is a null event independent of technology.

(ii) If $r < n$, the factor endowment of each country must belong to the cone spanned by $\phi_{\sigma_1}(q), \ldots, \phi_{\sigma_n}(q)$. If $K = 2$, we get the so-called "lens condition" illustrated in Fig.2, where the techniques are defined by a_1 and a_2 as in the case shown in Fig.1, but where we have an additional technique, for simplicity assumed to belong to the diagonal.

In this case, the factor price equalization domain will depend on commodity prices, or rather, on the amount of each commodity produced in the integrated equilibrium, which shows up in the lengths of the vectors $\phi_{\sigma_1}(q), \ldots, \phi_{\sigma_3}(q)$. This means that for the assessment of the probability of FPE, we need to take the commodity production into account.

We shall not consider the general case but restrict our attention to the case illustrated, where one of the techniques coincide with the diagonal. Then the length of the segment parallel to the diagonal parametrizes the output in the third technique, and given this length,

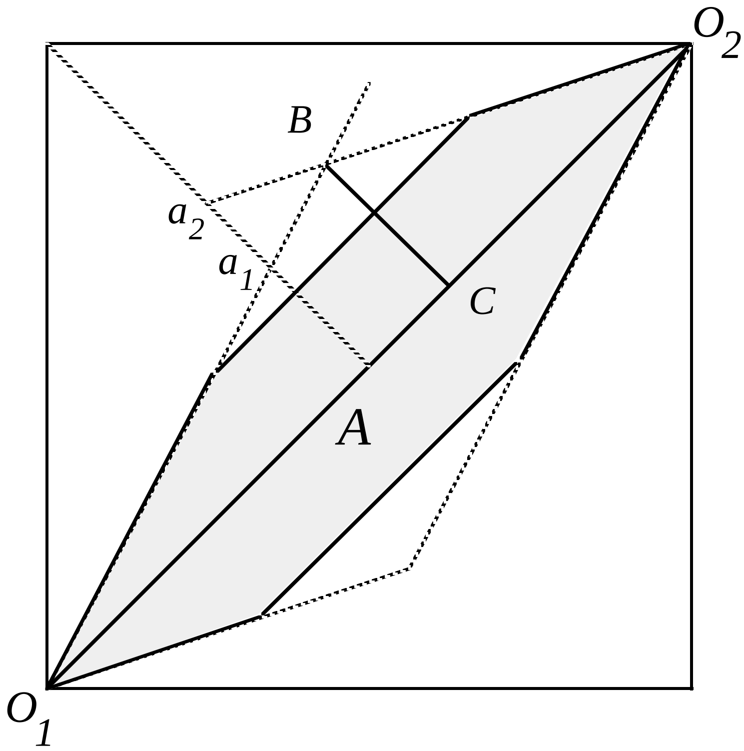

Figure 2

the factor price equalization domain is uniquely determined. Compared to the case of only 2 techniques, the area of the FPE domain is reduced by a triangle with top in B (and its symmetric counterpart). Letting t be the fraction of the distance from B to C which is not in the FPE domain we have that the probability of FPE, given that $a_3 = \left(\frac{1}{2}, \frac{1}{2}\right)$ can be found as

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{\frac{\sqrt{2}}{2}} \left[2\sqrt{2}\frac{\alpha\beta}{\alpha+\beta} - \int_0^1 \sqrt{2}t^2 \frac{2\alpha\beta}{\alpha+\beta}\, dt \right] d\alpha d\beta$$
$$= \frac{2}{3} \int_0^{\frac{\sqrt{2}}{2}} \int_0^{\frac{\sqrt{2}}{2}} 2\sqrt{2}\frac{\alpha\beta}{\alpha+\beta}\, d\alpha d\beta = \frac{1}{3},$$

showing that the impression obtained from the figure is sustained by the computation. It should of course be stressed that the numerical value of the probability of FPE has been established only for the special case where one of the techniques coincides with the diagonal.

5. Concluding remarks

In the present paper, we have considered a probabilistic approach to the phenomenon of factor price equalization, assessing the likelihood of its occurrence when country endowments and the underlying technologies are sampled in a random way. The main results obtained all point in the same direction, namely that factor price equalization is a rather unlikely event, with the probability of its occurrence going rapidly to zero when the number of commodities and factors increase.

As it has been mentioned above, results of this type may not contribute much in a direct way to our understanding of real life phenomena (where, however, equalization of factor prices has not been among the first consequences of globalization), but they may be helpful in our overall assessment of the models used to understand reality. Among these, the classical trade model has been around for half a century and is still very much alive. The concept of factor price equalization has been used also in extensions of this model to cover cases of imperfect competition, where most of the analysis pertains to situations where endowments are in the factor price equalization domain. The intuition to be obtained from the present work is that this type of analysis covers only a small part of what must be expected to occur, a very small part indeed if the results are to be generalized above the 2 by 2 case.

It might be argued, that the choices of uniform distributions and the restriction to Leontief technologies are restrictive and have little if any economic content, and this is largely true, but again the approach should be thought of as supplementing the textbook intuition obtained in the 2 by 2 case with something that applies to higher dimensions. This corresponds to the approach taken in geometric probability (see e.g. Santaló (1976)), where lines or convex bodies are sampled at random.

We have considered only the two-country case when deriving bounds for the probability of factor price equalization. It is easily seen that this probability does not increase when the number of countries grows. Indeed, for given technology the probability is bound by the relative area of the FPE domain to the power of K. Since increasing the number of commodities and factors is more relevant and has a spectacular effect, we have concentrated on this. It goes without saying that further and sharper bounds can be obtained by a more detailed analysis.

Appendix: The expected volume of a random subsimplex

In this section, we provide an assessment of P_n, the expected relative volume of a subsimplex of $\triangle_n$ the vertices of which are chosen at random. For the one-dimensional simplex $\triangle_2 = [0, 1]$ we have that

$$P_2 = \int_0^1 |b - a|\, da db = \int_0^1 \left[\int_0^b (b - a)\, da + \int_b^1 (a - b)\, da \right] db = \frac{1}{3}.$$

For the higher-dimensional versions, we need some preliminary considerations.

LEMMA 1. *Let $d \geq 2$ be arbitrary and let $T = \{x \in \mathbb{R}_+^d \mid \sum_{i=1}^d x_i \leq 1\}$. Consider the subsets $T(t_1, \dots, t_d) = \text{conv}(\{0, t_1 e_1, \dots, t_d e_d\})$ of T for $(t_1, \dots, t_d) \in [0,1]^d$. Then*

$$\int_0^1 \cdots \int_0^1 m_d(T(t_1, \dots, t_d))\, dt_1 \dots dt_d \leq \frac{1}{2} m_{d-1}(\text{conv}(\{e_1, \dots, e_d\}),$$

where m_d denotes Lebesgue measure in $\mathbb{R}^d$.

PROOF: Since $m(T(t_1, \dots, t_d)) \leq m(T(t_1, 1, \dots, 1))$, the assessment follows directly after integrating over t_1. $\square$

As is seen, a much sharper bound could be obtained, depending on the dimension d, but the present crude bound will suffice for our purposes.

LEMMA 2. *Let $n \geq 3$, and define P_n by*

$$P_n = \int_{a_1, \dots, a_n \in \triangle_n} \frac{m(\text{conv}(\{a_1, \dots, a_n\}))}{m(\triangle_n)}.$$

Then $P_n \leq \frac{1}{3} 2^{-(n-2)}$.

PROOF: By induction in n; for $n = 2$ the result was proved above. Assume that the lemma holds for all $2 \leq k < n$, write $\triangle_n = \text{conv}(\{e_1, \dots, e_n\})$ and consider an arbitrary subsimplex $D = \text{conv}(\{a_1, \dots, a_n\})$. Discarding cases where e_1 belongs to the affine subspace spanned by a facet of D, we may assume that there is some vertex in D, say a_1, such that $D \subset D' = \text{conv}(\{e_1, a_2 \dots, a_n\})$, and we restrict attention to subsimplices containing e_1.

Let $\hat{a}_i$, for $i = 2, \dots, n$, be the intersection of the rays from e_1 through a_i with the facet $\text{conv}(\{e_2, \dots, e_n\})$. Then using Lemma 1 and the fact that relative Lebesgue measure is invariant under linear maps, we get that

$$\frac{m(\text{conv}(\{e_1, a_2, \dots, a_n\}))}{m(\text{conv}(\{e_1, \hat{a}_2, \dots, \hat{a}_n\}))} \leq \frac{1}{2},$$

and it follows that

$$\frac{m(\text{conv}(\{e_1, a_2, \dots, a_n\})}{m(\triangle_n)} \leq \frac{1}{2} \frac{m(\text{conv}(\{\hat{a}_2, \dots, \hat{a}_n\})}{m(\text{conv}(\{e_2, \dots, e_n\})} \leq \frac{1}{2}\frac{1}{3} 2^{-(n-3)} = \frac{1}{3} 2^{-(n-2)},$$

where we have used the induction hypothesis. $\square$

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