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Optimizing Production under Uncertainty

Generalization of the State-Contingent Approach and Comparison with the EV Model

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Abstract

This Working Paper derives criteria for optimal production under uncertainty based on the state-contingent approach (Chambers and Quiggin, 2000), and discusses potential problems involved in applying the state-contingent approach in a normative context. The analytical approach uses the concept of state-contingent production functions and a definition of inputs including both sort of input, activity and allocation technology. It also analyses production decisions where production is combined with trading in state-contingent claims such as insurance contracts. The final part discusses the relative benefits and of using the state-contingent approach in a normative context, compared to the EV model.

Keywords: decision making, state-contingent, optimization, insurance contract, expected utility, utility function, production, input.

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Preface

This research work has been done by professor (docent) Svend Rasmussen.

Associate professor Lars Otto and head of division Mogens Lund have reviewed the paper. Secretary Inger Sommer has edited the final version.

Institute of Food and Resource Economics
Production and Technology Division, Marts 2006

Mogens Lund

1. Introduction

The classical approach to the problem of optimizing production under risk/uncertainty is the expected utility model (EU model). The EU-model is, in its basic form, relatively general. The tradition has developed over time that the EU-model is applied empirically as a model where utility is maximized as a function of the expected value and variance of profit (EV model), based on stochastic production functions (Dillon and Anderson, 1990; Hardaker et al., 1997; Robison and Barry, 1987).

This approach to decision making under uncertainty has been severely criticized by Chambers and Quiggin in their 2000 book on state-contingent production, and in subsequent papers. Their main problem is that the traditional approach typically does not consider the interaction between the uncontrolled (uncertain) variables and the decision variables controlled by the decision maker. Furthermore, although Dillon and Anderson (1990) realized the basic need for modelling this kind of interaction, they did not derive criteria for optimal production that went beyond maximizing utility, defined as a function of expected value and variance of profit.

In the state-contingent approach Chambers and Quiggin (2000) developed the foundation for alternative ways of describing and analysing production decisions under uncertainty. The state-contingent approach has the advantage that it explicitly considers the interaction between controllable inputs and uncontrolled inputs (the uncertain states of nature). In a recent article, Rasmussen (2003) used the state-contingent approach to derive criteria for optimal production (input use) under uncertainty. Criteria were derived for the one variable input case, as well as for different types of input, including state-specific and state-allocable¹⁾ input. He illustrates that the state-contingent approach has the merit of being based on well-known marginality principles and optimization tools, and indicates that the state-contingent approach has its own weaknesses in empirical application. Thus, the basic problem of not knowing the decision maker's utility function persists, and state-contingent production functions are typically not available. Therefore, the question of how to apply the theory of state-contingent production to real problems of decision making under uncertainty still has no clear answer.

¹⁾ A term first used by Chambers and Quiggin (2000).

The objectives of this article are to further develop and generalize the criteria for optimal production under uncertainty, and to discuss alternative procedures for application of the state-contingent approach in a normative context.

In the first part of the article, the criteria derived by Rasmussen are generalised to the multi-variable input case. It is shown that the output-cubical technology approach (Chambers and Quiggin, 2000: 53-54) is in fact an appropriate vehicle to use in a normative context²⁾, because it provides an operative way of handling the so-called state-allocable inputs. If inputs are defined as a combination of sort of input, activity, and application technology, then state-specific and state-allocable inputs are just special cases of any input defined this way. Specific criteria for these two types of inputs are therefore redundant; the general criteria derived will cover any type of input.

The first part of the article also demonstrates that optimal production decisions under uncertainty may be identified without knowledge of the state-contingent utility function, when there are markets for state-contingent insurance contracts. The integration of production and insurance decisions illustrates the analytical power of the state-contingent approach.

The second part of the article focuses on the problems related to the normative application of the state-contingent theory. The state-contingent approach is compared to the Expected Utility (EU) model, both with respect to choice of utility function and the description of production technology (production function). In this context, the differences between *state-contingent* and *stochastic* production functions are discussed, and it is proposed that in empirical contexts, it is appropriate to consider *state-contingent production functions* as being themselves *stochastic production functions*.

²) Despite that Chamber and Quiggin claim that it has “somewhat pathological properties” (Chambers and Quiggin, 2002a, p. 516).

2. General criteria for optimal production

Consider a producer who wants to optimize the production of one or more outputs. Both the production and the output prices are uncertain in the sense that yields and prices depend on uncertain future conditions called *states of nature*. The state of nature that determines yields and prices reveals itself only after application/allocation of inputs. Therefore, production decisions must be taken without knowing the future state of nature. The only thing known about the future state is that nature will pick one of S possible states of nature. Probabilities of each state of nature may or may not be known, but the decision maker holds - at least implicitly - expectations concerning the frequency with which each possible state of nature will prevail.

The decision-maker wants to maximize the utility function:

$$(1) \quad W(\mathbf{q}) = W(q_1, \dots, q_S)$$

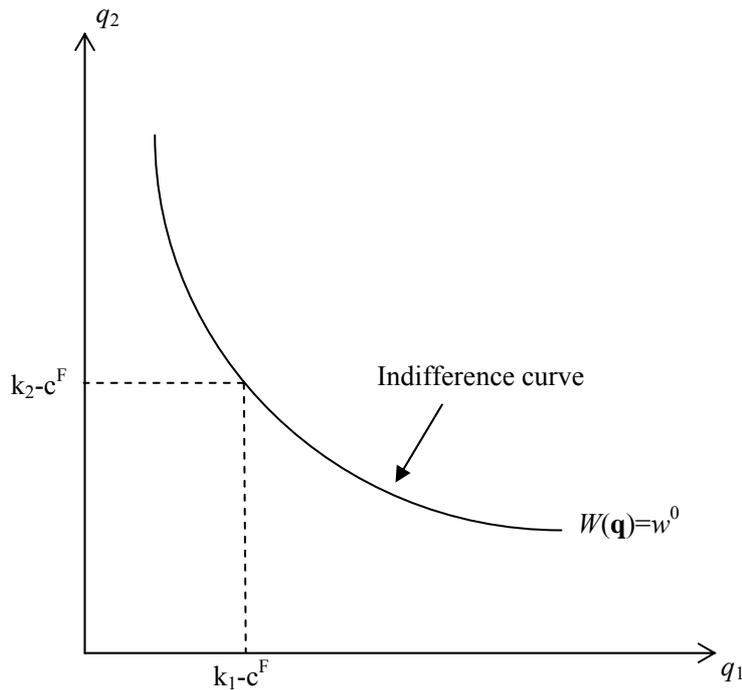
where W is a continuously differentiable non-decreasing, quasi-concave function, and $\mathbf{q}=(q_1, \dots, q_S)$ is a vector of net-incomes in the S possible states of nature, determined as:

$$(2) \quad q_s = \sum_{m=1}^M z_{ms} p_{ms} - \sum_{i=1}^n w_i x_i - c^F + k_s \quad (s=1, \dots, S)$$

where z_{ms} is production of output m in state s , p_{ms} is the price of output m in state s , x_i is the amount of variable input of sort i ($i=1, \dots, n$) used in the production of the M outputs, w_i is the price of input i ($i=1, \dots, n$), c^F is fixed costs, and k_s is a pre-determined state-contingent income from other sources in state s .

First consider the case in which no production takes place. In this case, the wealth is determined by the net-income vector $q_s = k_s - c^F$ ($s=1, \dots, S$), which is illustrated for $S=2$ in figure 1. Thus the utility without production is w^0 .

Figure 1. Wealth without production



To supplement the income $k_s - c^F$ ($s=1, \dots, S$), the decision-maker may carry out production. The production technology is, as a starting point, given in implicit form as a convex function $H: \mathfrak{R}_+^{M \times S + N} \rightarrow \mathfrak{R}$:

$$(3) \quad H(\mathbf{z}, \mathbf{x}) = H(z_{11}, \dots, z_{ms}, \dots, z_{MS}, x_1, \dots, x_N) \leq 0$$

where \mathbf{z} is a $M \times S$ matrix of state-contingent output of M products (z_{11}, \dots, z_{MS}) and \mathbf{x} is a vector of input (x_1, \dots, x_N), of which the first n elements are variable inputs, and the last $N - n$ elements are fixed inputs. The amount of fixed inputs is restricted by:

$$(4) \quad \sum_{m=1}^M x_{jm} - x_j^F \leq 0 \quad (j = n+1, \dots, N)$$

where x_{jm} is the amount of fixed input j allocated to production of output m and x_j^F is the amount of fixed input j .

If a budget restriction applies, then:

$$(5) \quad \sum_{i=1}^n w_i x_i - C^0 \leq 0$$

where C^0 is the given budget.

The production plan which maximizes utility in (1) is determined by the amount of variable inputs (x_1, \dots, x_n), the amount of outputs ($z_{11}, \dots, z_{ms}, \dots, z_{MS}$) and the allocation of the fixed inputs (x_{n+1}, \dots, x_N) which maximizes the Lagrangian:

$$(6) \quad L = W(q_1, \dots, q_S) - \mu H(\mathbf{z}, \mathbf{x}) - \sum_{j=n+1}^N \gamma_j \left(\sum_{m=1}^M x_{jm} - x_j^F \right) - \delta \left(\sum_{i=1}^n w_i x_i - C^0 \right)$$

where μ , γ_j and δ are Lagrange multipliers for the three restrictions (3), (4), and (5), respectively³.

Implicit in this decision making problem is also the problem of *how* to use each individual sort of input, i.e. which *activities* to perform and what *technology* (application *method* and the *timing* of application) to use.

This problem (of how to use the input) is typically *not* explicitly considered in non-stochastic models. The reason for this is that if a number of alternative activities or technologies are available, then it is implicitly assumed that the most efficient ones will be used.

However, in the state-contingent world, it is *not* possible to identify or even define unambiguously the most efficient technology and activities. Rather, this depends on the state of nature, which - according to the general assumption made earlier - only

³) To simplify, the following derivations only consider one output ($M=1$).

reveals itself *after* the production decision has been taken. For example, if nature picks the state s , then the most efficient way to apply a certain pesticide (sort of input) would have been with a spray-nozzle of *type a* (application method a). However, if nature picks the state t , the most efficient way to apply the pesticide would have been with a spray-nozzle of *type b* (application method b). Another example concerns *timing*. If nature picks the state u , then the most efficient time to apply a certain type of fertilizer would have been April 1st. If nature picks state v instead, the most efficient time for fertilizer application might be June 1st. Finally, consider the following example concerning choice of *activity*. If nature picks the state p (drought), the most efficient way to use a fixed amount of an input “effort”, would be to perform the activity of building irrigation facilities. If nature picks state q (heavy rain) instead, then the most efficient activity would be to build flood-control. (This last example demonstrates that the term *state-allocable inputs* (Chambers and Quiggin, 2000: 39 and Rasmussen 2003: 459) is in fact a special case of the more general problem of choice of activity and technology in the case of state-contingent production).

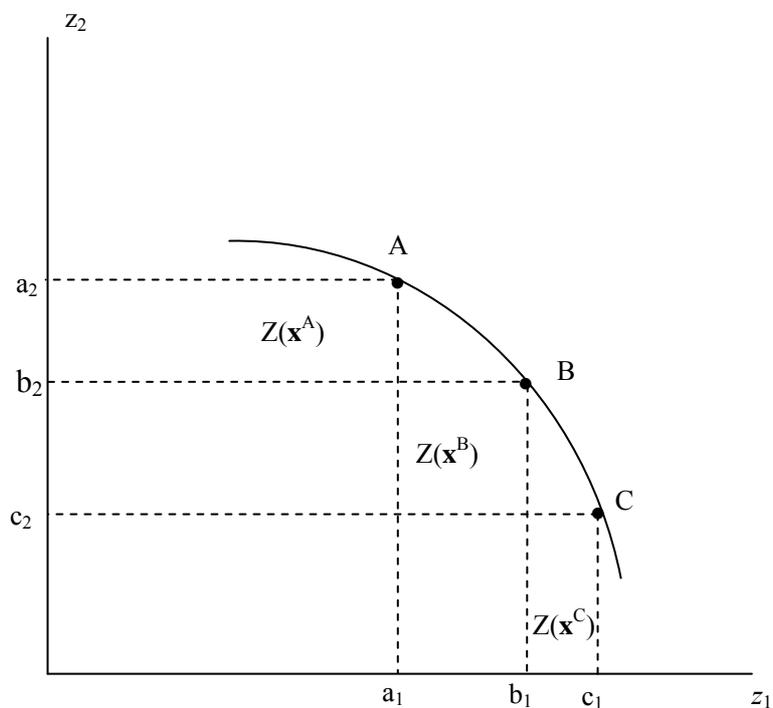
The state-contingent approach to describing production decisions under uncertainty therefore adds another dimension to the decision making problem: Is it necessary not only to determine the optimal procurement of variable inputs, but also the allocation of both variable and fixed inputs to alternative activities, and to the appropriate technology to use.

One way to integrate all three decision dimensions into the optimisation problem described in (1) - (6) above is to consider each input x_i ($i=1, \dots, N$) as being a specific combination of *sort of input* (nitrogen fertilizers, pesticide, labour, fuel, etc.), *activity*, and *technology*. Thus (using the examples from above): a pesticide applied with a spray-nozzle of *type a* is considered a separate input from the same pesticide applied with a spray-nozzle of *type b*; a certain sort of fertilizer applied April 1st is considered a different input from the same sort of fertilizer applied June 1st; and “effort” applied to building irrigation facilities is considered a different input from “effort” applied to building flood control. In the following, this definition of inputs will be used unless stated otherwise.

If H (in (3)) is a continuously differentiable function with non-vanishing derivatives, then the conditions for optimal production may be derived from (6). However, with the definition of inputs used here, the *technology is output-cubical* involving non-

substitutability between state contingent outputs⁴⁾, and these conditions therefore do *not* apply.

Figure 2. Substitution between states of nature



Rather, when the *technology is output cubical* non-decreasing and quasi-concave state-contingent production functions $f_s(\mathbf{x})$ ($s=1, \dots, S$) exist where the output set Z is:

⁴⁾ In this context, the term ‘non-substitutability between state contingent outputs’ is an analytical formality. In reality, substitutability between state-contingent outputs still exists (*ex ante*) in the sense that the decision maker may choose between different input vectors. This is illustrated for $S=2$ in Figure 2, where the rectangles with the corners A, B and C are the state-contingent output sets for the input vectors \mathbf{x}^A , \mathbf{x}^B , and \mathbf{x}^C , respectively, and where the curve through A, B, and C is the state-contingent product transformation curve (see Chambers and Quiggin, 2000: 40-41 and 67).

$Z(\mathbf{x}) = \{z: z_s \leq f_s(\mathbf{x}), s=1, \dots, S\}$ and the production function is $f_s(\mathbf{x}) = \max\{z_s\}$ (See Chambers and Quiggin, 2000: 54).

Thus, the production technology $H(z, \mathbf{x})$ in (3) may be expressed as:

$$(7) \quad H_s(z_s, \mathbf{x}) = z_s - f_s(\mathbf{x}) \leq 0 \quad (s=1, \dots, S)$$

and the Lagrangean function in (6) becomes⁵⁾:

$$(8) \quad L = W(q_1, \dots, q_S) - \sum_{s=1}^S \mu_s H_s(z_s, \mathbf{x}) - \gamma \left(\sum_{j=n+1}^N x_j - x^F \right) - \delta \left(\sum_{i=1}^n w_i x_i - C^0 \right)$$

where q_s ($s=1, \dots, S$) is defined in (2). (Further notice, that there are $N-n$ different ways (different activities or technologies) in which the amount of the fixed input x^F may be used).

Differentiating (8) with respect to x_i , ($i = 1, \dots, N$) and z_s , ($s = 1, \dots, S$) assuming interior solutions yield the first order conditions listed under the following three headings I, II, and III :

I. Optimal combination of variable inputs:

$$(9) \quad \frac{w_i}{w_j} = \frac{\sum_{t=1}^S \frac{\partial W}{\partial q_t} VMP_{ti}}{\sum_{t=1}^S \frac{\partial W}{\partial q_t} VMP_{tj}} \quad (i, j = 1, \dots, n)$$

where VMP_{ti} is $p_t(\partial f_t / \partial x_i)$ i.e. the Value of Marginal Product of input x_j in state t .

If one assumes risk neutrality, then (9) reduces to:

$$(10) \quad \frac{w_i}{w_j} = \frac{E(VMP_i)}{E(VMP_j)} \quad (i, j = 1, \dots, n)$$

⁵⁾ To simplify, (8) and the following derivations consider only one fixed input.

which expresses that for optimal production, the risk-neutral decision maker should combine variable inputs in such a way that the ratio of the expected marginal products is equal to the price ratio.

II. Optimal application of variable inputs:

The general condition for optimal application of variable input is:

$$(11) \quad \sum_{t=1}^S p_t \frac{\partial W}{\partial q_t} \frac{\partial f_t}{\partial x_i} = w_i \left(\sum_{t=1}^S \frac{\partial W}{\partial q_t} + \delta \right) \quad (i = 1, \dots, n)$$

which under risk-neutrality reduces to:

$$(12) \quad E(VMP_i) = w_i(1 + \delta) \quad (i = 1, \dots, n)$$

(12) shows that a risk neutral decision maker should continue to add variable input, as long as the expected value of the marginal product is higher than the input price, subject to any budgetary restriction.

III. Optimal allocation of fixed inputs:

The general condition for optimal allocation of a fixed input is:

$$(13) \quad \sum_{t=1}^S p_t \frac{\partial W}{\partial q_t} \frac{\partial f_t}{\partial x_j} = \gamma \quad (j = n+1, \dots, N)$$

Under risk-neutrality (13) reduces to:

$$(14) \quad E(VMP_j) = \gamma \quad (j = n+1, \dots, N)$$

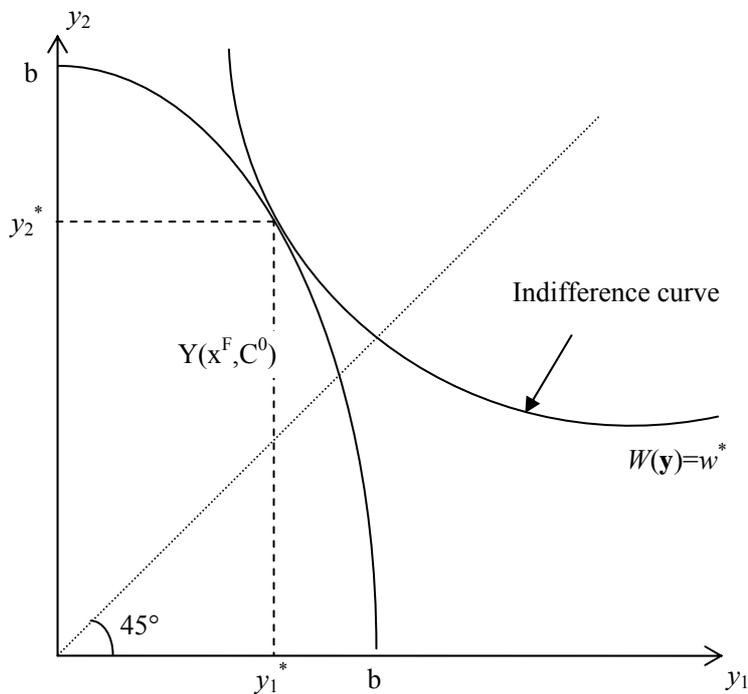
Thus, fixed input that may be used in $N-n$ different ways (different activities or technologies) should be allocated between these different activities/technologies so that the expected values of the marginal products are equalized across activities/technologies. In the earlier example of allocating “effort” between building irrigation and flood-control, condition (14) states that the last unit of “effort” should yield the same expected economic outcome in both activities.

The result of optimizing production, as derived in (9) – (14), is illustrated in figure 3 for $S=2$. The origin of the system of coordinates in figure 3 corresponds to $k_s - c^F$ ($s=1, \dots, S$) in Figure 1, so that the axes in figure 3 measure *changes in income* compared to the no production alternative illustrated in figure 1. Thus, the *net-returns* y_s on the axes in figure 3 are estimated as (compare with (2)):

$$(15) \quad y_s = z_s p_s - \sum_{i=1}^n w_i x_i \quad (s=1, \dots, S)$$

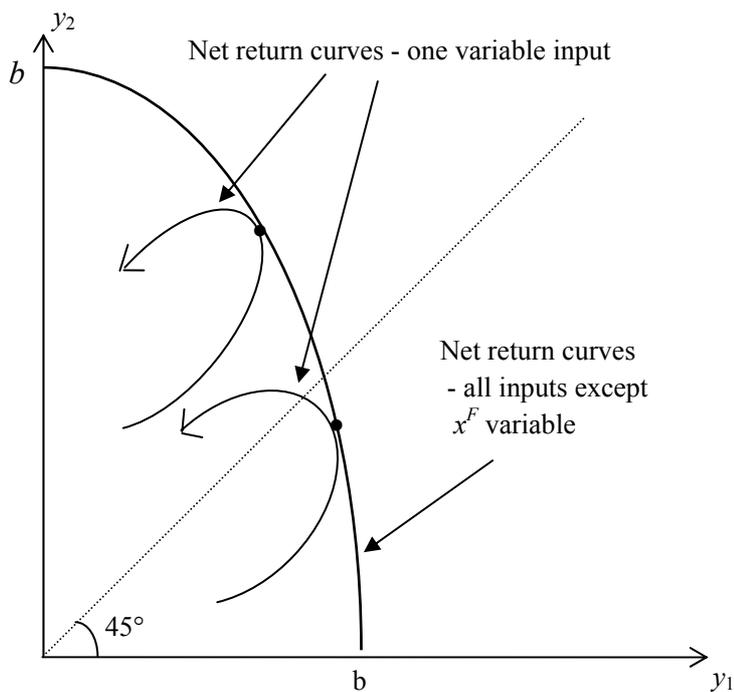
and the optimal production plan (assuming that the set of net-returns Y is convex) is the production plan that yields the state-contingent net-returns (y_1^*, y_2^*) .

Figure 3. Optimal state-contingent income



The net return set $Y(x^F, C^0)$ (the y 's below the curve bb in figure 3) is determined by the amount of fixed input (x^F) and the budget (C^0). To interpret this set, compare with the net-return curve derived in Rasmussen (2000: 466-467) for the one-variable-input case. In figure 4, two such one-variable-input net return curves are illustrated with an increasing amount of one variable input in the direction of the arrow, assuming that all other inputs are fixed. The two curves illustrate different amounts of the other inputs except x^F , which is the same (fixed) in the two cases.

Figure 4. Derivation of net return curve



The net return curve bb may now be interpreted as the envelope curve for all such possible one-variable-input curves, of which only two are shown in figure 4. Thus, the implicit assumption behind the net return possibility curve bb is that all inputs are used efficiently.

3. Challenges to empirical application

The criteria derived for optimal production ((11) and (13)) involve, not only the derivatives of the state-contingent production functions, but also the derivatives of the state-contingent utility function. To implement the criteria derived, i.e. to use the criteria in decision making contexts, or to perform comparative static analysis, knowledge of these functions is required.

Elicitation of the utility function has historically been one of the major problems encountered in application of the Expected Utility (EU) model⁶. However, the state-contingent approach also requires the identification of risk preferences. And while the endeavours in the literature have focused on elicitation of von Neumann-Morgenstern (NM) utility functions, a more general preference structure on which the state-contingent approach may be based entails further challenges.

There are cases in which empirical application is possible without explicit knowledge of the utility function. These exist if there are what Hirshleifer and Riley (1992: 51) call Complete Contingent Markets (CCM), i.e. markets for direct trading in state-contingent claims such as markets for insurance, or Complete Asset Markets (CAM), i.e. markets for assets including financial assets such as loans, futures, and options. In the current context, these markets may be used to re-allocate state-contingent incomes. If such markets (and therefore prices of state-contingent incomes) exist, then it is possible to separate the production decision and the consumption decision (Hirshleifer and Riley, 1992, p.56).

This is illustrated by expanding the earlier decision problem (1)-(5) to include the possibility of buying insurance contracts. Consider S state-specific insurance contracts, each of which gives an indemnity of 1 € in state s , and which have a price of v_s ($s=1, \dots, S$). The decision maker may buy any number of the S contracts. If k_s is the number of state s contracts ($s=1, \dots, S$), then the decision maker faces the following (extended) optimization problem (compare with (8)):

$$(16) \quad L = W(q_1, \dots, q_S) - \sum_{s=1}^S \mu_s H_s(z_s, \mathbf{x}) - \gamma \left(\sum_{j=n+1}^N x_j - x^F \right) - \delta \left(\sum_{i=1}^n w_i x_i + \sum_{s=1}^S v_s k_s - C^0 \right)$$

⁶) Moschini and Hennesy (2001) give a good review of the published research on identifying risk preferences (the NM-utility function) in the EU-model context.

where the net-income q_s in state s is now (compare with (2)):

$$(17) \quad q_s = \sum_{m=1}^M z_{ms} p_{ms} - \sum_{i=1}^n w_i x_i - c^F - \sum_{t=1}^S v_t k_t + k_s \quad (s=1, \dots, S)$$

where the last two terms are the net income from the insurance activity. Notice that unlike in (2), k_s ($s=1, \dots, S$) are now decision variables).

This extended problem specification yields the following additional first order conditions (assuming interior solutions):

$$(18) \quad \frac{v_s}{v_t} = \frac{\frac{\partial W}{\partial q_s}}{\frac{\partial W}{\partial q_t}} \quad (i, j = 1, \dots, n)$$

so that insurance activities should be combined in such a way that the ratio between marginal utility in any two states is equal to their price ratio. Further, if the budget restriction is excluded ($\delta=0$), then by inserting the first order condition:

$$(19) \quad \frac{\partial W}{\partial q_s} = v_s \left(\frac{\partial W}{\partial q_1} + \dots + \frac{\partial W}{\partial q_S} \right) = v_s \sum_{t=1}^S \frac{\partial W}{\partial q_t} \quad (s=1, \dots, S)$$

into the original first order condition (11) for determining the optimal amount of input (without insurance), then (11) becomes:

$$(20) \quad \sum_{s=1}^S p_s v_s \frac{\partial f_i}{\partial x_i} = w_i \quad (i = 1, \dots, n)$$

This condition shows that in the presence of the opportunity to buy insurance contracts, the allocation of variable input may be separated from the consumption decision, because the utility function does not enter in (20). The same is true for the fixed input (insert (19) in (13)). Thus, the production decision has been separated from the consumption decision.

This result is not new in itself, but it has not been previously stated explicitly in a simple operational form such as (20). Further, it provides a good example of the potential analytical power of the state-contingent approach.

Although insurance options will not be further discussed here, it is clearly an increasingly important approach (Allen and Lueck, 2002; Babcock and Hennessy, 1996; Chambers and Quiggin, 2002b; Chambers and Quiggin, 2000; Coble and Knight, 2002; Pannell et al., 2000; Quiggin, 2002). As a basis for normative application, the approach using direct or indirect markets for state-contingent claims, appears more promising than those using effort to elicitate the decision-maker's utility function. In the ideal case of insurable occurrences and actuarially fair insurance contracts, the problem of optimizing production boils down to the problem of identifying the production plan that maximizes the expected net-return (Nelson and Loehman, 1987).

Because the objective of this paper is to compare the state-contingent approach with the empirical approaches typically taken in the EU-model context (Hardaker et al., 1997), subsequent analysis will be based on the direct approach, i.e. without considering markets for state-contingent claims. In that case, empirical application involves choosing/estimating an appropriate utility function.

3.1. Choice of Utility Function.

The utility function in the EU-model

The popular choice of functional form of the NM utility function in the EU model framework is the negative exponential:

$$(21) \quad v(y) = 1 - e^{-\lambda y}$$

where λ ($\lambda > 0$) is the Arrow-Pratt coefficient of absolute risk aversion. Although this form implies the assumption of constant absolute risk aversion (CARA), which is *not* usually regarded as a desirable property (Hardaker et al., 1997), it has found extensive use in applied analyses of decision making under uncertainty (Allen and Lueck, 2002; Chavas and Holt, 1990; Pope and Just, 1991; Smith et al., 2003) due to its mathematical/analytical properties: if y is normally distributed, then the expected utility is a simple function of expected value (E) and Variance (V), i.e.

$$W(\mathbf{y}) = E(y) - \frac{\lambda}{2} V(y).$$

Other desirable functional forms are the logarithmic (which has decreasing absolute risk aversion (DARA)):

$$(22) \quad v(y) = \ln(y)$$

and the power function:

$$(23) \quad v(y) = \frac{1}{1-r} y^{(1-r)}$$

where r is the Arrow-Pratt coefficient of relative risk aversion, i.e. $r = y\lambda$. Like the logarithmic, the power function also has decreasing absolute risk aversion (DARA) and in addition has constant relative risk aversion (CRRA). The quadratic function $v(y) = y - by^2$ ($b > 0$) has also proved popular, because it implies an EV utility function, i.e. $W(\mathbf{y}) = E(y) - hV(y) - h[E(y)]^2$.

The properties of the different types of utility functions may be illustrated by deriving the *rate of substitution in utility* of y_s for y_t (RSU_{st}), defined as the absolute value of the slope $-dy_s/dy_t$ of an iso-utility curve (indifference curve as in Figure 3) in state-space⁷. Thus, a utility function $W(\mathbf{y})$ based on the NM-utility function in (21) has the following property:

$$(24) \quad RSU_{st} \equiv \frac{\partial W / \partial y_s}{\partial W / \partial y_t} = \frac{\pi_s}{\pi_t} e^{-\lambda(y_s - y_t)}$$

where π_s ($s=1, \dots, S$) is the probability of state s , while a utility function based on the NM-utility function (22) has the property:

$$(25) \quad RSU_{st} \equiv \frac{\partial W / \partial y_s}{\partial W / \partial y_t} = \frac{\pi_s y_t}{\pi_t y_s}$$

and a utility function based on (23) has the property:

⁷) See Dillon and Anderson (1990), p. 125 where this term is used to describe the slope in EV-space.

$$(26) \quad \text{RSU}_{st} \equiv \frac{\partial W / \partial y_s}{\partial W / \partial y_t} = \frac{\pi_s}{\pi_t} \left(\frac{y_t}{y_s} \right)^r$$

It follows from (24), (25), and (26) that as the marginal rate of substitution (i.e. the amount of state s income that is substituted for one unit of state t income) increases, the greater becomes the difference in income in the two states of nature.

The utility function in the state-contingent model

In the state-contingent framework, utility functions based on the EU-model may still be applied because Expected Utility is just a special case of the more general utility function. However, it is appropriate to consider more general functional forms and in this context the restrictions to place on the utility function.

The most common restriction to place on preferences is that they are convex. This implies quasi-concavity of the utility function over stochastic incomes, i.e. the decision maker is risk-neutral or risk-averse. It should be noted that utility functions based on the Neumann-Morgenstern utility functions mentioned in the previous section all fulfil this restriction. In the general case of the state-contingent framework, preferences depend only on the state-contingent outcomes, and not explicitly on the probabilities as is the case in the EU-model.

Aside from the linear utility function (in which case the utility is simply the expected value of net returns), the simplest functional form describing risk aversion in the state-contingent framework is the Cobb-Douglas:

$$(27) \quad W(\mathbf{y}) = a_0 \prod_{t=1}^S y_t^{a_t}$$

where $0 < a_t < 1$ ensures that the function is quasi-concave in y_t .

Because the relative probabilities are given as the slope of the indifference curve along the bisector (Chambers and Quiggin, 2000: 90), i.e.:

$$(28) \quad \frac{\pi_s}{\pi_t} = \frac{\partial W / \partial y_s}{\partial W / \partial y_t} \Big|_{y_1 = \dots = y_s = y} \quad (s, t \in \Omega)$$

where Ω is the set of possible states of nature, then the relative probabilities are:

$$(29) \quad \frac{\pi_s}{\pi_t} = \frac{a_s}{a_t} \quad (s, t \in \Omega)$$

Thus, the choice of the parameters of the Cobb-Douglas utility function ($a_t, t=1, \dots, S$) is also the choice of the relative (subjective) probabilities implicitly attached to the different states of nature. On the other hand, if the probabilities π_s ($s = 1, \dots, S$) have already been determined, then the relative value of the parameters a_s ($s = 1, \dots, S$) are determined by (29).

A Cobb-Douglas utility function has the derivatives:

$$(30) \quad \frac{\partial W}{\partial y_s} = \frac{a_s}{y_s} W(\mathbf{y}) \quad (s \in \Omega)$$

and therefore the RSU_{st} is:

$$(31) \quad RSU_{st} \equiv \frac{\partial W / \partial y_s}{\partial W / \partial y_t} = \frac{\pi_s}{\pi_t} \frac{y_t}{y_s}$$

Comparing (31) with (25), reveals that a Cobb-Douglas utility function provides the same marginal rate of substitution (slope of the indifference curve) as an EU utility function, based on the logarithmic form of the NM utility function (22).

Equation (31) also shows that the Cobb-Douglas utility function features constant relative risk aversion (CRRA) (the expansion path is a straight line through origin) and therefore decreasing absolute risk aversion (DARA), which according to Meyer (2002) is an acceptable assumption. However, the Cobb-Douglas function differs from the EU model in the sense that the marginal utility of income in state s (see (30)) depends not only on the relative probability of state s (a_s) and of the net return in state

$s(y_s)$, but also on the net return in the other states of nature ($W(\mathbf{y})$). In this sense, even the relatively simple Cobb-Douglas function potentially provides more flexibility in the description of preferences than the utility functions based on the popular NM forms mentioned above.

3.2. The Production Function

The EU-model and the state-contingent model are based on different approaches to describing the technology. While the EU-approach focuses on *the stochastic production function* and an estimation of probability distributions for yield (and prices), the state-contingent approach focuses on *state-contingent production functions*, and therefore yields (and prices) contingent on discrete states of nature.

Just (Just, 2003) compares the EU-model and the state-contingent model. He argues that the relative advantage of the two approaches depends on the number of moments of the probability distribution it is necessary to estimate, compared to the number of states of nature. In particular, if there are many states of nature then the state-contingent approach is weakened and "...most distributions facing farmers have large numbers of potential outcomes (states of nature)" (p.140). As examples, most yield and price distributions have a large number of outcomes: depending on the units used for measuring yields and prices there may even be thousands of yields and prices, and a correspondingly large number of states to consider.

Although at first sight this appears important, it also exposes potential errors in comparing the two approaches. While the EU-model typically focuses on the probability distributions of yields and prices (i.e. the *consequences* of the uncertain environment), the state-contingent approach focuses directly on the uncertain environment (i.e. the *states of nature*). Thus, yields and prices are not (as indicated by Just) "states of nature", but rather *consequences* of states of nature. In a decision making context, the state-contingent approach is appropriate, because it treats explicitly the realized yield of a crop of wheat as a *consequence*, not only of the controllable inputs (the input vector \mathbf{x}), but also of the *interaction* with the non-controllable inputs, i.e. the "states of nature" (amount of rain, hours of sunshine, etc.). However, it is not evident how to approach empirically a problem involving perhaps thousands of discrete states of nature that require estimation of a state-contingent production function usable within the above theoretical framework for each state of nature.

The stochastic production function

The EU-model is typically based on what Chambers and Quiggin (2002a) call *Stochastic Production Functions*, i.e. functions of the type $z = f(\mathbf{x}, \varepsilon)$. The empirical problem relate to estimating the function and the probability density function of the error term ε , (or at least the first two or three moments)⁸. The Just-Pope production function form $z = g(\mathbf{x}) + h(\mathbf{x})\varepsilon$ (Just and Pope, 1978) has proven especially popular in applied analyses (Moschini and Hennessy, 2001), and has been used by for instance Larson *et al.* (2002), Smith *et al.* (2003), and Horowitz and Lichtenberg (1994)).

The state-contingent production function

In the state-contingent model, one of the immediate problems for applied work is the definition of the possible states of nature. A state of nature is formally defined as a complete description of the external conditions (the environment) in the sense that, given a specific state of nature (non-controllable inputs) and a production decision (amount of controllable inputs), the consequences (outputs or prices) are uniquely determined (there is no error term).

Consider first the ideal case where the complete state-space is the set $\Omega = \{1 \dots S\}$, and where the applied researcher has available all the S *state-contingent production functions*:

$$(32) \quad z_s = f_s(\mathbf{x}) \quad (s \in \Omega)$$

either in the form of mathematical functions, or in discrete data form.

Although in this case, application of the state-contingent model is straightforward, in practice this (ideal) situation rarely occurs. A state may be quantified by a vector of *state-variables* describing the state-space using quantitative variables such as temperature, sunshine, precipitation, etc. While one can easily imagine a state description being *complete* in the sense that everything relevant has been described/registered, this is typically *not* the case in empirical applications because either not all the state-

⁸) In the EU approach, much energy is used in choosing a type of distribution (Normal, beta, etc) and estimating the parameters, typically the expected value and the variance (Dillon and Anderson, 1990; Goodwin and Ker, 2002).

variables influencing output or the values of the relevant state-variables are known. In both cases the state description is *incomplete*, and it should be noted that the state-variables may be uncertain due to measurement error.

In empirical contexts, the state-contingent output (the output given a specific registered⁹⁾ state of nature) is typically a stochastic variable. Thus the available state-contingent production functions are themselves stochastic production functions, i.e.:

$$(33) \quad z_s = f_s(\mathbf{x}, \varepsilon) \quad (s \in \Omega^E)$$

where ε is a stochastic error term given state s , and Ω^E is the set of states for which production functions have been estimated ($\Omega^E \subseteq \Omega$).

Thus, the typical situation facing the applied researcher is that if state-contingent production functions are available at all, they refer to only a few possible (real) states of nature. For those real states for which they are available, the state-description is probably incomplete, i.e. it has the general form of a stochastic production function shown in (33).

This will be the main obstacle to applying the state-contingent approach in an empirical/normative context. Experimental data and farm response data typically do not provide the information necessary to estimate state-contingent production functions. The question is therefore, how the potential advantages of the state-contingent approach may be exploited, when the data necessary to support the approach are not available. Further research is necessary to answer this question.

It is interesting to note that the *stochastic production function* and the *state-contingent production function* are *special cases* of the more general description of the technology in (33). In the special case that $\Omega^E = \Omega$ (production functions have been estimated for *all* possible states), then the error term in (33) vanishes, and the technology description is in the form of (non-stochastic) state-contingent production functions as in (32). In the special case that $\Omega^E = \{1\}$ (the production functions refers to no specific state), then the model (33) reduces to the (pure) stochastic production function.

⁹⁾ We use the term *registered state* to describe the way in which a state is actually (empirically) registered. If not all relevant state-variables are registered or if the registered level of the individual state-variables is uncertain, then the state description is *incomplete*. A *real state* is the actual state, which exists independently of being registered or not. In the following when we use the term *state*, we mean *registered state* unless explicitly stated.

4. Conclusion

In this paper I have derived criteria for optimal application of variable and fixed input in the multiple input - one output case, based on the state-contingent approach. It has been shown that with the output-cubical technology as the basic model, any type of input may be analysed within the general model framework developed. The analysis has also shown that under uncertainty (the state-contingent approach), the decision on *how* inputs are used (choice of activities and technology), has to be considered explicitly, because the relative efficiency of alternative activities/technologies may vary across states of nature.

Production decisions may be combined with other (risk management) activities. In this paper, it has been shown that by introducing the option of trading in state-contingent claims (insurance contracts), the production decision can be separated from the consumption decision. This result is not new, but underlines the potential analytical power of the state-contingent approach compared to the classical EU model.

Applications of the criteria derived require that state-contingent production functions and utility functions based on state-contingent income measures are known. Because most empirical work concerning optimizing production under uncertainty has historically been based on the expected utility model, the approach based on the state-contingent approach carries with it new challenges with respect to both modelling utility and choice of functional forms and procedures for estimating state-contingent production functions. In the paper it is shown that even relatively simple functional forms of the utility function based on state-contingent income measures involve a higher degree of flexibility in describing preferences than popular functional forms applied in the EU framework do. Concerning production technology, the state-contingent production function and the normally applied stochastic production function have been compared, and it is shown that the two ways of describing the production technology are just special cases of a more general description: a stochastic, state-contingent production function.

It is unrealistic to expect that production functions may be estimated for all possible states of nature, and indeed state-contingent production functions may be estimated for only a few states of nature. The main conclusion concerning empirics is that when this is the case, each of the state-contingent production functions available should be considered, being a stochastic production function.

This raises the question of the relative merit of the state-contingent approach and the EV model for empirical application in a normative context. While the state-contingent approach has clear advantages if state-contingent production functions are available for all states of nature, it is not clear whether this is the case if one has available (or is able to estimate) only a few *stochastic*, state-contingent production functions. It is proposed that this question be further investigate (e.g. using Monte-Carlo simulation).

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