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1 The first-passage time distribution for 2 the diffusion model with variable drift

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23 **Abstract**

24 The Ratcliff diffusion model is now arguably the most widely applied model for response time
25 data. Its major advantage is its description of both response times and the probabilities for
26 correct as well as incorrect responses. The model assumes a Wiener process with drift between
27 two constant absorbing barriers. The first-passage times at the upper and lower boundary
28 describe the responses in simple two-choice decision tasks, for example, in experiments with
29 perceptual discrimination or memory search. In applications of the model, a usual assumption is
30 a varying drift of the Wiener process across trials. This extra flexibility allows accounting for slow
31 errors that often occur in response time experiments. So far, the predicted response time
32 distributions were obtained by numerical evaluation as analytical solutions were not available.
33 Here, we present an analytical expression for the cumulative first-passage time distribution in
34 the diffusion model with normally distributed trial-to-trial variability in the drift. The solution is
35 obtained with predefined precision, and its evaluation turns out to be extremely fast.

36 **Keywords**

37 Diffusion model; Response time modeling

38 Background

39 The diffusion model for response times was proposed about 40 years ago (Ratcliff, 1978) as a
40 continuous-time, continuous-state generalization of earlier discrete-time random walk models
41 (Laming, 1968; Link & Heath, 1975). One of its major advantages over standard response time
42 (RT) analyses (i.e., comparison of mean RTs) is the simultaneous analysis of both response time
43 and accuracy. This avoids problems of speed-accuracy trade-offs that are possible confounders
44 of the results and generally difficult to interpret (e.g., Pachella, 1974).

45 The standard diffusion model assumes a Wiener process with drift ν and diffusion
46 coefficient σ^2 (typically fixed either at $\sigma^2 = 1$ or $\sigma^2 = 0.01$ because it only scales the other
47 parameters) evolving over time in the presence of two absorbing barriers (located at 0 and
48 $a > 0$). Each barrier is associated with one response alternative. The barriers can be viewed as
49 response criteria, that is, the distribution of the first passage time to either barrier produces the
50 predicted response times distribution for the response alternative associated with the barrier.

51 Although the model is well motivated and the approach is appealing, two issues remain
52 that are often seen as major obstacles for a wider application of the model. Firstly, there is no
53 closed-form solution available for the partial differential equation (PDE) of a diffusion process
54 with the necessary boundary conditions. The available solutions (e.g., Feller, 1968) all require
55 the evaluation of infinite series. These series can be shown to converge quite quickly (Navarro &
56 Fuss, 2009; Blurton, Kesselmeier, & Gondan, 2012; Gondan, Blurton, & Kesselmeier, 2014).
57 However, when fitting the model to data, the series has to be evaluated over and over again,
58 which may take a considerable amount of time. This is especially true if more general versions
59 of the model are fitted to data (see next section). In that case, several numerical integrations
60 have to be carried out that are associated with their own (possibly unknown) approximation

61 errors. However, for parameter estimation it is useful to have an exact result to avoid numerical
 62 problems during estimation (e.g., rough likelihood surfaces).

63 Secondly, the available solutions only cover the standard Wiener process with constant
 64 drift across trials. By analogy to the signal detection model (Tanner & Swets, 1954) and based
 65 on common sense arguments (the “resonance” metaphor), Ratcliff (1978) argued that the drift
 66 rate v shows inter-trial variability that can be described by a normal distribution: $v \sim N(v, \eta^2)$.
 67 For example, one direct consequence of this assumption is that in a response signal paradigm,
 68 perceptual sensitivity d' asymptotes and does not reach infinity with signal time t (Ratcliff,
 69 1978, Eq. 10). However, this extra variability comes at the cost of a missing analytical form for
 70 the model predictions. Hence, model predictions must be obtained by numerical evaluation
 71 instead (Ratcliff & Tuerlinckx, 2002). Interestingly, the *density* function¹ is known for the case of
 72 normally distributed drift rates (e.g., Horrocks & Thompson, 2004) and it has been used in the
 73 past for fitting the diffusion model to response time data (Ratcliff & Tuerlinckx, 2002; Wiecki,
 74 Sofer, & Frank, 2013). For the lower barrier, it is

$$76 \quad g(t | v, \eta^2, a, w) = \frac{1}{\sqrt{t^3(1+\eta^2t)}} \exp \left[\frac{-v^2t - 2vaw + \eta^2(aw)^2}{2(1+\eta^2t)} \right] \sum_{j=0}^{\infty} (-1)^j r_j \phi \left(\frac{r_j}{\sqrt{t}} \right) \quad (1)$$

77
 78 where $r_j = ja + aw$ for even j or $r_j = ja + a(1 - w)$ for odd j , and $\phi(x)$ denotes the standard
 79 normal density function evaluated at x , and $0 < w < 1$ is the relative starting point of the
 80 Wiener process between the two barriers. Without loss of generality the diffusion coefficient σ^2

¹ Note that the distribution (density) is technically not a probability distribution (density) but a defective distribution (density) because it does not integrate to unity. One obtains a proper distribution (density) by summing the distributions (densities) from the upper and lower criteria or by normalizing through the respective absorption probability.

81 has been omitted in (1), as $g'(t | v, \eta^2, \sigma^2, a, w) = g(t | v/\sigma, \eta^2/\sigma^2, a/\sigma, w)$. The density
82 function is useful if maximum likelihood estimation is desired. However, if parameter estimates
83 are to be obtained from binned data, for example by chi-square methods (e.g., Ratcliff & Smith,
84 2004) or by the quantile maximum likelihood method (Heathcote, Brown, & Mewhort, 2002)
85 one must rely on numerical integration of the first-passage time density to obtain the
86 distribution function.

87 Since its introduction additional parameters for inter-trial variability have been added to
88 the model (Ratcliff & Rouder, 1998; Ratcliff & Tuerlinckx, 2002). Thus, the “full” Ratcliff diffusion
89 model fit now requires the numerical evaluation of three integrals (see Tuerlinckx, 2004, Eq. 3).
90 This can become time consuming as the computational complexity raises exponentially
91 (Tuerlinckx, 2004) and all these integrals must be evaluated on infinite series.

92 Here, we present an analytical solution for the first-passage time distribution of the
93 Ratcliff (1978) model with drift variation. The solution is of theoretical interest and especially for
94 applications of the model. For the application, it increases speed and establishes a pre-defined
95 accuracy of the fitting procedure. It is readily available for use in existing software packages like
96 DMAT (Vandekerckhove & Tuerlinckx, 2008). Researchers that have implemented or seek to
97 implement their own fitting routines will also benefit from the solution as it guarantees a
98 computationally efficient computation with accuracy up to some pre-defined level.

99 **The cumulative distribution function for the Ratcliff diffusion model**

100 Recently, Gondan and colleagues (2014) reported a solution of the PDE for a Wiener process
101 with constant drift between two absorbing barriers that is using a representation stated in
102 terms of the Mills ratio (Hall, 1997). We would like to remind the reader of some of the

103 favorable properties of this representation. Firstly, it is numerically very stable and no numerical
 104 problems arise during the calculation of the infinite series. Secondly, and contrasting its related
 105 representation (e.g., Blurton et al., 2012), it is defined for all real drift rates and does not suffer
 106 from a singularity at zero drift. Clearly, this is very important when integrating over drift rates.
 107 Thirdly, it gives the distribution function and not the survivor function so that the separate
 108 calculation of the overall absorption probability at a specific barrier is not necessary. In the most
 109 widely adapted representation of the first-passage time cumulative distribution, the survivor
 110 function is used. In that case, the series must be subtracted from the probability of terminating
 111 at the associated barrier to obtain the cumulative distribution (see Ratcliff, 1978, Eq. A12 and
 112 p. 105f, for the motivation of this approach). Obtaining the cumulative directly avoids problems
 113 in the derivation regarding this probability with drift variation over trials (see Tuerlinckx, 2004).
 114 Apart from the latter issue, these points also hold for the alternative solution that is available
 115 and usually used in fitting the diffusion model (Ratcliff, 1978; Ratcliff & Tuerlinckx, 2002).
 116 However, the analytic solution for this CDF with inter-trial variability in drift rates is yet
 117 unknown.

118 Using the aforementioned representation (1), the cumulative distribution function $F(t)$
 119 of the first-passage time of a Wiener process with drift v between two absorbing barriers placed
 120 at 0 and $a > 0$ and starting at aw ($0 < w < 1$) to the lower boundary can be expressed by the
 121 infinite series (Hall, 1997)

122

$$123 \quad F(t | v, a, w) = \exp\left(-vaw - \frac{v^2 t}{2}\right) \sum_{j=0}^{\infty} (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \left[M\left(\frac{r_j - vt}{\sqrt{t}}\right) + M\left(\frac{r_j + vt}{\sqrt{t}}\right) \right] \quad (2)$$

124

125 with r_j and $\phi(x)$ as defined above, and $M(x) = \frac{1-\Phi(x)}{\phi(x)}$ denoting the inverse hazard function
 126 (the ‘‘Mills ratio’’) for the standard normal distribution.

127 In order to obtain a solution for the more general process with trial-to-trial variability in
 128 drift rate v , one must seek a solution of the integral $\int \psi(x) \cdot F(t | x, a, w) dx$, that is, one must
 129 integrate over the density $\psi(x)$ of the assumed drift distribution and the first-passage time
 130 distribution $F(t)$. Because drift rates can take any real value and due to the correspondence
 131 with the signal detection model (Tanner & Swets, 1954), the normal distribution is usually
 132 chosen as a possible distribution for the drift rates (Ratcliff, 1978, Eqs. 8, A24, & A25). Thus, we
 133 replace $\psi(x)$ by the normal density $\phi(x | v, \eta^2)$ with mean v and variance η^2 . Let
 134 $G(t | v, \eta^2, a, w)$ be the first-passage time distribution of such a process,

$$135$$

$$136 \quad G(t | v, \eta^2, a, w) := \int_{-\infty}^{\infty} \phi(x | v, \eta^2) \cdot F(t | x, a, w) dx$$

$$137 \quad = \int_{-\infty}^{\infty} \phi(x | v, \eta^2) \exp\left(-xaw - \frac{x^2 t}{2}\right) \sum_{j=0}^{\infty} (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \left[M\left(\frac{r_j - xt}{\sqrt{t}}\right) + M\left(\frac{r_j + xt}{\sqrt{t}}\right) \right] dx$$

138

139 The series is absolutely convergent (see Appendix A) so that summation and integration can be
 140 exchanged and we may write

$$141$$

$$142 \quad G(t | v, \eta^2, a, w) = \sum_{j=0}^{\infty} g_j(t | v, \eta^2, a, w)$$

143

144 with

$$145$$

$$146 \quad g_j := (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \int_{-\infty}^{\infty} \exp\left(-xaw - \frac{x^2 t}{2}\right) \phi(x | v, \eta^2) \left[M\left(\frac{r_j - xt}{\sqrt{t}}\right) + M\left(\frac{r_j + xt}{\sqrt{t}}\right) \right] dx.$$

147 Each term of the series is composed of two summands, so for simplicity let us define

148

$$149 \quad g_j^- := (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \int_{-\infty}^{\infty} \exp\left(-xaw - \frac{x^2 t}{2}\right) \phi(x | \nu, \eta^2) M\left(\frac{r_j - xt}{\sqrt{t}}\right) dx$$

150

151 and

152

$$153 \quad g_j^+ := (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \int_{-\infty}^{\infty} \exp\left(-xaw - \frac{x^2 t}{2}\right) \phi(x | \nu, \eta^2) M\left(\frac{r_j + xt}{\sqrt{t}}\right) dx.$$

154

155 with $g_j = g_j^- + g_j^+$ (we omitted the arguments for notational compactness). We first derive g_j^- .

156 Replacement of Mills ratio and application of $1 - \Phi(x) = \Phi(-x)$ leads to

157

$$158 \quad g_j^- = \frac{(-1)^j}{\sqrt{2\pi\eta^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\nu)^2}{2\eta^2} - xaw - \frac{x^2 t}{2}\right] \exp\left(-\frac{r_j^2}{2t}\right) \Phi\left(\frac{xt - r_j}{\sqrt{t}}\right) \exp\left[\frac{(r_j - xt)^2}{2t}\right] dx.$$

159

160 Then, simplification and rearrangement according to powers of x results in

161

$$162 \quad g_j^- = \frac{(-1)^j}{\sqrt{2\pi\eta^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2\eta^2} + \left(\frac{\nu}{\eta^2} - aw - r_j\right)x - \frac{\nu^2}{2\eta^2}\right] \Phi\left(x\sqrt{t} - \frac{r_j}{\sqrt{t}}\right) dx.$$

163

164 For convenience, we define $p := \frac{\nu}{\eta^2} - aw - r_j$. Next, by completing the square one obtains

165

$$166 \quad g_j^- = \frac{(-1)^j}{\sqrt{2\pi\eta^2}} \exp\left(-\frac{\nu^2}{2\eta^2} + \frac{\eta^2}{2} p^2\right) \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x}{\eta} - \eta p\right)^2\right] \Phi\left(x\sqrt{t} - \frac{r_j}{\sqrt{t}}\right) dx$$

167 The required integral is of the form $\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(\delta x - \gamma)^2\right] \Phi(\beta x - \alpha) dx$, to which the
 168 solution is $\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(\delta x - \gamma)^2\right] \Phi(\beta x - \alpha) dx = \frac{\sqrt{2\pi}}{\delta} \left[1 - \Phi\left(\frac{\alpha\delta - \gamma\beta}{\sqrt{\beta^2 + \delta^2}}\right)\right]$ (see Appendix B).
 169 With the obvious correspondence of α , β , γ , and δ , this leads to

$$170$$

$$171 \quad g_j^- = (-1)^j \exp\left(-\frac{v^2}{2\eta^2} + \frac{\eta^2}{2} p^2\right) \Phi\left(\frac{\eta p \sqrt{t-r_j/\eta^2 t}}{\sqrt{t+1/\eta^2}}\right)$$

$$172 \quad = (-1)^j \exp\left[\frac{\eta^2}{2}(aw + r_j)^2 - v(aw + r_j)\right] \Phi\left[\frac{vt - \eta^2(aw + r_j)t - r_j}{\sqrt{t(1 + \eta^2 t)}}\right]$$

173 Similarly,

$$174$$

$$175 \quad g_j^+ = (-1)^j \exp\left[\frac{\eta^2}{2}(aw - r_j)^2 - v(aw - r_j)\right] \Phi\left[\frac{vt - \eta^2(aw - r_j)t + r_j}{\sqrt{t(1 + \eta^2 t)}}\right].$$

176
 177 By combining the results for g_j^- and g_j^+ , we get $g_j(t | v, \eta^2, a, w)$ of the series $G(t | v, \eta^2, a, w)$ as
 178 the required analytical solution. However, we further develop the result to obtain a
 179 representation using the Mills ratio again because of its favorable numerical properties (see
 180 above).

$$181$$

$$182 \quad g_j^- = (-1)^j \exp\left[\frac{\eta^2}{2}(aw + r_j)^2 - v(aw + r_j)\right] \left\{1 - \Phi\left[-\frac{vt - \eta^2(aw + r_j)t - r_j}{\sqrt{t(1 + \eta^2 t)}}\right]\right\}$$

$$183 \quad = \frac{(-1)^j}{\sqrt{2\pi}} \exp\left[\frac{-v^2 t - 2vaw + \eta^2(aw)^2}{2(1 + \eta^2 t)}\right] \exp\left[-\frac{r_j^2 + \eta^2 t r_j^2}{2t(1 + \eta^2 t)}\right] M\left[\frac{r_j - vt + \eta^2(aw + r_j)t}{\sqrt{t(1 + \eta^2 t)}}\right]$$

$$184 \quad = (-1)^j \exp\left[\frac{-v^2 t - 2vaw + \eta^2(aw)^2}{2(1 + \eta^2 t)}\right] \phi\left(\frac{r_j}{\sqrt{t}}\right) M\left[\frac{r_j - vt + \eta^2(aw + r_j)t}{\sqrt{t(1 + \eta^2 t)}}\right].$$

185

186

187 Similarly, we have

188

$$189 \quad g_j^+ = (-1)^j \exp \left[\frac{-v^2 t - 2vaw + \eta^2 (aw)^2}{2(1+t\eta^2)} \right] \phi \left(\frac{r_j}{\sqrt{t}} \right) M \left[\frac{r_j + vt + \eta^2 (r_j - aw)t}{\sqrt{t(1+\eta^2 t)}} \right].$$

190

191 The cumulative distribution function then reads as

192

$$193 \quad G(t | v, \eta^2, a, w) = \exp \left[\frac{-v^2 t - 2vaw + \eta^2 (aw)^2}{2(1+\eta^2 t)} \right] \times$$

$$194 \quad \sum_{j=0}^{\infty} (-1)^j \phi \left(\frac{r_j}{\sqrt{t}} \right) \left\{ M \left[\frac{r_j - vt + \eta^2 (r_j + aw)t}{\sqrt{t(1+\eta^2 t)}} \right] + M \left[\frac{r_j + vt + \eta^2 (r_j - aw)t}{\sqrt{t(1+\eta^2 t)}} \right] \right\}. \quad (3)$$

195

196 This is the analytic result of the model proposed by Ratcliff (1978). The absorption
 197 probability at the upper barrier is obtained by $G(t | -v, \eta^2, a, 1 - w)$. For non-unit variance σ^2 ,
 198 $G'(t | v, \eta^2, \sigma^2, a, w) = G(t | v/\sigma, \eta^2/\sigma^2, a/\sigma, w)$. The above solution is interesting in several
 199 aspects. Firstly, it bears similarities with the already known density function (Eq. 1) and the
 200 solution for an unrestricted Wiener process with normally distributed drift (Ratcliff, 1978, Eq. 8).
 201 Secondly, for $\eta^2 = 0$, it simplifies to the distribution function $F(t | v, a, w)$ of a standard Wiener
 202 process (Eq. 2) with constant drift v . In other words, it can be safely used in a fitting routine,
 203 regardless of the (empirical) question, whether there is inter-trial variability in the data or not. If
 204 no such variation is observed, the function safely converges to the no-variation case.

205 Convergence

206 Because the r_j are strictly increasing, and the Mills ratio is strictly decreasing in its argument,
 207 the function $F(t | v, a, w)$ in (2) is a strictly decreasing alternating series (Gondan et al., 2014). A

208 similar argument can be made for (3): Because $G(t \mid \nu, \eta^2, a, w)$ is a weighted sum of different
 209 $F(t \mid \nu, a, w)$, it is a strictly decreasing alternating series as well, so that its evaluation can be
 210 stopped as soon as the first summand g_J is below some pre-defined error tolerance $\varepsilon > 0$.
 211 Then, it is guaranteed that the truncation error—that is, the difference between the true
 212 distribution (3) and the truncated series evaluated up to some J —is not greater than the pre-
 213 defined tolerance level.

214 If a reasonable estimate for the number of required terms is known, the precision of the
 215 truncated solution is improved (e.g., by aggregating terms in increasing order). The number of
 216 required terms can be obtained by solving, for example, $g_{2K} \leq \varepsilon$ for even $J = 2K$. We first note
 217 that for sufficiently large r_{2K} (such that the argument of ϕ is positive), a simple upper bound
 218 $h_{2K} \geq g_{2K}$ is found with

$$220 \quad h_{2K} = 2 \exp \left[\frac{-\nu^2 t - 2\nu a w + \eta^2 (a w)^2}{2(1 + \eta^2 t)} \right] \times \phi \left[\frac{r_{2K} - |\nu| t}{\sqrt{t(1 + \eta^2 t)}} \right] M \left[\frac{r_{2K} - |\nu| t}{\sqrt{t(1 + \eta^2 t)}} \right]$$

$$221 \quad = 2 \exp \left[\frac{-\nu^2 t - 2\nu a w + \eta^2 (a w)^2}{2(1 + \eta^2 t)} \right] \left\{ 1 - \Phi \left[\frac{r_{2K} - |\nu| t}{\sqrt{t(1 + \eta^2 t)}} \right] \right\}$$

222
 223 The inequality $h_{2K} \leq \varepsilon$ is then solved for $J = 2K$,

$$225 \quad J \geq \frac{\sqrt{t(1 + \eta^2 t)}}{a} \cdot \Phi^{-1} \left\{ 1 - \frac{1}{2} \exp \left[\frac{\nu^2 t + 2\nu a w - \eta^2 (a w)^2}{2(1 + \eta^2 t)} + \log \varepsilon \right] \right\} + \frac{|\nu| t}{a} - w. \quad (4)$$

226 Positivity of the arguments of ϕ is given for $J \geq \frac{|\nu| t}{a} - w$.

227 **Efficiency**

228 The CDF in (3) can readily be used for parameter estimation in combination with a fitting
229 function that relies on the CDF—such as chi-square methods or the quantile maximum
230 likelihood estimation (Heathcote et al., 2002). Our first analyses using the solution on simulated
231 data showed that it can be readily used with reasonable computational effort (Table 1): The
232 number of terms needed for convergence up to a pre-defined tolerance ε is generally very low.
233 The number of terms mainly depends on the barrier separation parameter a and the time t at
234 which the function is evaluated: Similar to the constant drift case (Eq. 2), larger t and smaller a
235 lead to slower convergence of the series. The other parameters ν , η^2 , and w have hardly any
236 influence on the convergence behavior. Because no numerical integration is required, a
237 tolerance of ε of approximately 1.5×10^{-8} seems appropriate (i.e., around the square root of
238 the smallest positive 32 bit floating-point number ε for which 1 is distinguishable from $1 + \varepsilon$).
239 With this tolerance, none of the calculations shown in Table 1 needed more than ten terms to
240 converge. It is also turned out that the upper bound for J (Eq. 4) is overly conservative. In any
241 case, the scenario in Table 1 is rather pessimistic as we assumed decision times up to 1200 ms
242 and $G(t | \nu, \eta^2, a, w)$ converges even quicker for lower values of t .

Table 1

Number of terms needed to achieve pre-defined accuracy.

η^2	Parameter		Number of terms	
	a	w	J from Eq. 4	Needed
0.01	0.08	.375	15	8
		.500	15	8
	0.11	.375	11	5
		.500	11	6
	0.14	.375	9	5
		.500	9	4
0.04	0.08	.375	23	7
		.500	23	8
	0.11	.375	17	5
		.500	17	5
	0.14	.375	13	4
		.500	13	4
0.09	0.08	.375	31	7
		.500	31	7
	0.11	.375	23	5
		.500	23	5
	0.14	.375	18	4
		.500	18	4

Note—Scaling parameter was set to $\sigma^2 = 0.01$. The table shows the number of terms needed to achieve accuracy $\varepsilon = 1.5 \times 10^{-8}$ at the lower barrier. The mean drift rate was also varied, $\nu \in \{0, \pm 0.1, \pm 0.2, \pm 0.3\}$, and the highest number was chosen. Time t was varied between 0.1 and 1.2 s; the values presented are for evaluation at 1.2 s as lower t generally lead to faster convergence.

243 Discussion

244 In this note we presented an analytical solution to the two-barrier diffusion model proposed by
 245 Ratcliff (1978). The solution is easily implemented (see online appendix) and allows for efficient
 246 and accurate calculation of the first-passage time CDF of a Wiener process with normally
 247 distributed drift rates across trials. The accuracy benefits of an analytic solution and except for

248 the truncation error which can be controlled for, no further inaccuracies occur in the calculation
249 of model predictions. With regard to the efficiency of the calculation we consider the provided
250 solution to lie between the computationally very efficient, but theoretically limited EZ-Diffusion
251 model (Wagenmakers, van der Maas, & Grasman, 2007) and packages like fastDM (Voss & Voss,
252 2007) and DMAT (Vandekerckhove & Tuerlinckx, 2007, 2008) which allow for a fit of the “full”
253 Ratcliff diffusion model with all the other mixture parameters (variable starting point, variable
254 residual component). The EZ-Diffusion model is computationally very efficient but uses only
255 small portions of the data; namely, mean and variance as well as the proportion of correct
256 responses. But it is computationally extremely efficient as explicit formulae of method of
257 moment estimators exist for the standard case without inter-trial variability. The solution
258 offered in this paper utilizes the full distribution and allows for trial-to-trial variation in drift
259 rates. The additional assumptions of trial-to-trial variation in residual (i.e., non-decision) time
260 (T_{er}) and starting point $z = aw$ could be added based the solution presented in this paper. This
261 additional variation requires numerical evaluation of two integrals—which should be
262 considerably faster than three integrals. Our solution is thereby fully compatible with the DMAT
263 toolbox (Vandekerckhove & Tuerlinckx, 2007). It would be interesting to see how performance
264 of DMAT improved if the provided solution was implemented. In any case, it should greatly
265 improve the accuracy and the speed of the estimation in self-written implementations (e.g., a
266 hierarchical Bayesian model of the Ratcliff diffusion model, see Wiecki et al., 2013).

267 **Conclusions**

268 Despite the obvious advantages of employing a computational model for response time and
269 response accuracy (Smith & Ratcliff, 2004), psychologists have—for a long time—only

270 reluctantly employed formal models (e.g., the two-barrier diffusion model). Recently, there has
271 been a surge in interest for the diffusion model and this article aims at further improving its
272 computational and numerical basis. The analytical solution guarantees a fast and accurate
273 calculation of model predictions for a diffusion model with normally distributed drift rates.

274 **Supplementary material**

275 The online supplement includes R (R core team, 2016) and Matlab code for Equation (3).

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329 Appendix A: Exchangeability of summation and integration

330 To exchange the integration and summation operators, one must show the absolute
331 convergence of the series. First, consider the series

332

$$333 \quad H(t, x \mid v, \eta^2, a, w) = \sum_{j=0}^{\infty} h_j(t, x \mid v, \eta^2, a, w)$$

334

335 with

336

$$337 \quad h_j := (-1)^j \exp\left(-xaw - \frac{x^2 t}{2}\right) \phi(x \mid v, \eta^2) \phi\left(\frac{r_j}{\sqrt{t}}\right) \left[M\left(\frac{r_j - xt}{\sqrt{t}}\right) + M\left(\frac{r_j + xt}{\sqrt{t}}\right) \right].$$

338

339 That is, $G(t \mid v, \eta^2, a, w) = \int_{-\infty}^{\infty} H(t, x \mid v, \eta^2, a, w) dx$. To establish exchangeability of

340 integration and summation, we use the ratio test to prove absolute convergence. The ratio of

341 consecutive terms in the series is:

342

$$343 \quad \left| \frac{h_{j+1}}{h_j} \right| = \left| \frac{\phi\left(\frac{r_{j+1}}{\sqrt{t}}\right) \left[M\left(\frac{r_{j+1} - xt}{\sqrt{t}}\right) + M\left(\frac{r_{j+1} + xt}{\sqrt{t}}\right) \right]}{\phi\left(\frac{r_j}{\sqrt{t}}\right) \left[M\left(\frac{r_j - xt}{\sqrt{t}}\right) + M\left(\frac{r_j + xt}{\sqrt{t}}\right) \right]} \right|.$$

344

345 Then, $\lim_{j \rightarrow \infty} \left| \frac{h_{j+1}}{h_j} \right| = Q < 1$ is a sufficient condition for absolute convergence. Hence, we must

346 show that

347

$$348 \quad \lim_{j \rightarrow \infty} \left| \frac{h_{j+1}}{h_j} \right| = \lim_{j \rightarrow \infty} \left[\frac{\phi\left(\frac{r_{j+1}}{\sqrt{t}}\right)}{\phi\left(\frac{r_j}{\sqrt{t}}\right)} \cdot \frac{M\left(\frac{r_{j+1} - xt}{\sqrt{t}}\right) + M\left(\frac{r_{j+1} + xt}{\sqrt{t}}\right)}{M\left(\frac{r_j - xt}{\sqrt{t}}\right) + M\left(\frac{r_j + xt}{\sqrt{t}}\right)} \right] < 1.$$

349 because $\phi(u), M(u) > 0$ for $u \in \mathbb{R}$. The arguments of the Mills ratio depend on x which may
 350 range from positive to negative infinity. Hence, we will not seek an explicit solution for the
 351 second factor. However, we know from the (log-) convexity of the Mills ratio for the standard
 352 normal distribution (Baricz, 2008) that $M(u)$ is strictly decreasing in $u \in \mathbb{R}$. As $r_{j+1} > r_j$ for all j ,
 353 we can conclude that the limit exists and that it is between zero and one:

354

$$355 \quad 0 \leq \lim_{j \rightarrow \infty} \frac{M\left(\frac{r_{j+1}-xt}{\sqrt{t}}\right) + M\left(\frac{r_{j+1}+xt}{\sqrt{t}}\right)}{M\left(\frac{r_j-xt}{\sqrt{t}}\right) + M\left(\frac{r_j+xt}{\sqrt{t}}\right)} \leq 1.$$

356

357 It remains to show convergence of the ratio of normal densities:

358

$$359 \quad \lim_{j \rightarrow \infty} \frac{\phi\left(\frac{r_{j+1}}{\sqrt{t}}\right)}{\phi\left(\frac{r_j}{\sqrt{t}}\right)} = \lim_{j \rightarrow \infty} \frac{\exp(-r_{j+1}^2)}{\exp(-r_j^2)}$$

360

361 The arguments of the normal density function do not depend on x . Because the r_j are
 362 differently defined for odd and even j , we must derive the limit for both cases. For
 363 simplification, use the compact notation $w' = 1 - w$. Assume that j is even and that $j + 1$ is
 364 odd, thus, $r_j = ja + aw$ and $r_{j+1} = (j + 1)a + aw'$. Then,

365

$$366 \quad \lim_{j \rightarrow \infty} \frac{\exp(-[(j+1)a+aw']^2)}{\exp(-(ja+aw)^2)} = \lim_{j \rightarrow \infty} \frac{\exp(-[a^2(j+1)^2+2(j+1)a^2w'+a^2w'^2])}{\exp(-[(ja)^2+2ja^2w+(aw)^2])}$$

$$367 \quad = \lim_{j \rightarrow \infty} \exp(-2ja^2(1+w'-w) - a^2[1+w'+w'^2-w^2])$$

$$368 \quad = 0,$$

369

370 since $w, w' \in (0, 1)$. For the alternative case, that is, j is odd and $j + 1$ is even, exchange w'

371 with w which does not change the result. Consequently, $\lim_{j \rightarrow \infty} \left| \frac{h_{j+1}}{h_j} \right| = 0 < 1$.

372 **Appendix B: Derivation of the definite integral**

373 As stated in the text, we seek a solution of the integral

374

$$375 \quad I(\alpha, \beta, \gamma, \delta) := \int_{-\infty}^{\infty} \exp[-(\delta x - \gamma)^2/2] \Phi(\beta x - \alpha) dx,$$

376

377 that is, a parametric function which suggests a solution by differentiation under the integral

378 sign:

379

$$380 \quad \frac{d}{d\alpha} I(\alpha, \beta, \gamma, \delta) = \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \exp[-(\delta x - \gamma)^2/2] \Phi(\beta x - \alpha) dx =$$

$$381 \quad (-1) \int_{-\infty}^{\infty} \exp[-(\delta x - \gamma)^2/2] \phi(\beta x - \alpha) dx$$

382

383 Replacing $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ and simplification yields:

384

$$385 \quad I_{d\alpha}(\alpha, \beta, \gamma, \delta) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{\beta^2 + \delta^2}{2} \left(x - \frac{\gamma\delta + \alpha\beta}{\beta^2 + \delta^2}\right)^2 - \frac{(\alpha\delta - \gamma\beta)^2}{2(\beta^2 + \delta^2)}\right] dx$$

386

387 Integration with respect to x gives

388

$$\begin{aligned}
389 \quad I_{d\alpha}(\alpha, \beta, \gamma, \delta) &= -\frac{1}{\sqrt{\beta^2 + \delta^2}} \exp\left[-\frac{(\alpha\delta - \gamma\beta)^2}{2(\beta^2 + \delta^2)}\right] \int_{-\infty}^{\infty} \phi\left(\frac{x - \frac{\gamma\delta + \alpha\beta}{\beta^2 + \delta^2}}{\frac{1}{\beta^2 + \delta^2}}\right) dx \\
390 \quad &= -\frac{1}{\sqrt{\beta^2 + \delta^2}} \exp\left[-\frac{(\alpha\delta - \gamma\beta)^2}{2(\beta^2 + \delta^2)}\right] = -\sqrt{2\pi} \phi\left(\frac{\alpha\delta - \gamma\beta}{\sqrt{\beta^2 + \delta^2}}\right).
\end{aligned}$$

391

392 Then, the indefinite integral with respect to α is given by:

393

$$394 \quad I(\alpha, \beta, \gamma, \delta) = -\sqrt{2\pi} \int \phi\left(\frac{\alpha\delta - \gamma\beta}{\sqrt{\beta^2 + \delta^2}}\right) d\alpha = -\frac{\sqrt{2\pi}}{\delta} \Phi\left(\frac{\alpha\delta - \gamma\beta}{\sqrt{\beta^2 + \delta^2}}\right) + C.$$

395

396 To obtain C , we may note that by definition of $I(\alpha, \beta, \gamma, \delta)$ it holds that397 $\lim_{\alpha \rightarrow \infty} I(\alpha, \beta, \gamma, \delta) = 0$. Thus,

398

$$399 \quad \lim_{\alpha \rightarrow \infty} \left[-\frac{\sqrt{2\pi}}{\delta} \Phi\left(\frac{\alpha\delta - \gamma\beta}{\sqrt{\beta^2 + \delta^2}}\right) + C \right] = 0 \Leftrightarrow -\frac{\sqrt{2\pi}}{\delta} + C = 0 \Leftrightarrow C = \frac{\sqrt{2\pi}}{\delta}$$

400

401 Finally, we have that

402

$$403 \quad I(\alpha, \beta, \gamma, \delta) = \frac{\sqrt{2\pi}}{\delta} \left[1 - \Phi\left(\frac{\alpha\delta - \gamma\beta}{\sqrt{\beta^2 + \delta^2}}\right) \right].$$

404

405 This solution is a more general version of a known result (for $\delta = 1$ and $\gamma = 0$, the above

406 solution corresponds to Eq. 10,010.8 in Owen, 1980).

Online supplement: R code

Tested with R version 3.3.1

```
# Distribution at Lower barrier (Eq. 3 of the article)
# t: time (vector)
# nu: average drift
# eta2: variance of the drift distribution
# sigma2: variance of Wiener process
# a: upper barrier
# w: relative position of  $X(0) = z$ ,  $w = z/a$ 
# eps: required precision
#
G_0 = function(t=1.2, nu=0.1, eta2=0.01, sigma2=0.01, a=0.08, w=.375,
  eps=sqrt(.Machine$double.eps))
{
  nu = nu / sqrt(sigma2)
  a = a / sqrt(sigma2)
  eta2 = eta2 / sigma2
  sqt = sqrt(t)
  sqet = sqt * sqrt(1 + eta2*t)
  G = numeric(length(t))
  j = 0
  repeat
  {
    rj = j*a + a*w
    logphi = dnorm(rj/sqt, log=TRUE)
    logM1 = logMill((rj - nu*t + eta2*(rj + a*w)*t) / sqet)
    logM2 = logMill((rj + nu*t + eta2*(rj - a*w)*t) / sqet)
    gj = exp(logphi + logM1) + exp(logphi + logM2)
    G = G + gj
    if(all(gj < eps))
      return(exp((-nu*nu*t - 2*nu*a*w + eta2*a*a*w*w)/2/(1 + eta2*t)) * G)

    j = j + 1
    rj = j*a + a*(1-w)
    logphi = dnorm(rj/sqt, log=TRUE)
    logM1 = logMill((rj - nu*t + eta2*t*(rj + a*w)) / sqet)
    logM2 = logMill((rj + nu*t + eta2*t*(rj - a*w)) / sqet)
    gj = exp(logphi + logM1) + exp(logphi + logM2)
    G = G - gj
    j = j + 1
  }
}
```



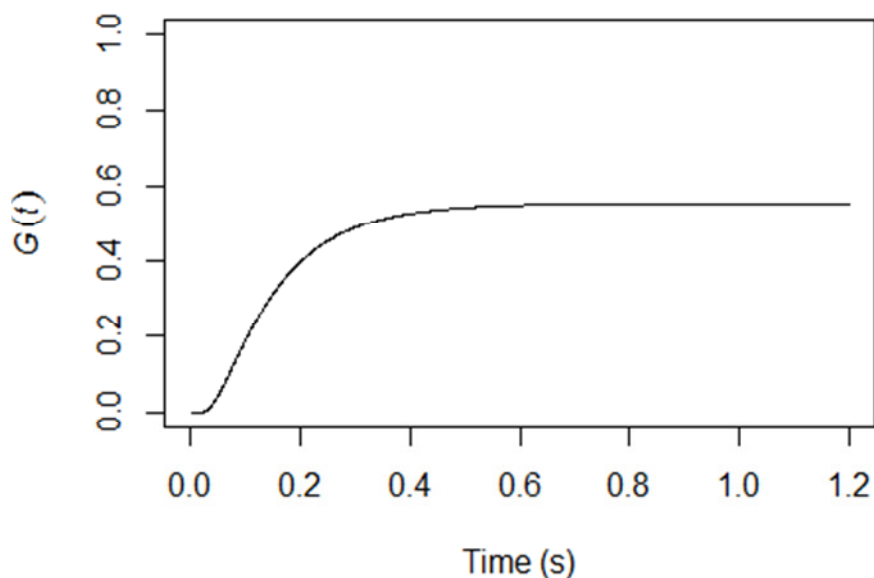
```

# Distribution at upper barrier
#
G_a = function(t=1.2, nu=0.1, eta2=0.01, sigma2=0.01, a=0.08, w=.375,
  eps=sqrt(.Machine$double.eps))
{
  G_0(t, -nu, eta2, sigma2, a, 1-w, eps)
}

# Log of Mill's ratio for the normal distribution
#
logMill = function(x) # Log of Mill's ratio
{
  m = numeric(length(x))
  m[x >= 10000] = -log(x[x >= 10000]) # Limiting case for x -> Inf
  m[x < 10000] = pnorm(x[x < 10000], lower=FALSE, log=TRUE) -
    dnorm(x[x < 10000], log=TRUE)
  m
}

# Example
#
plot(seq(0.001, 1.200, 0.001),
  G_a(t=seq(.001, 1.200, .001), nu=0.1, eta2=.01, sigma2=.01, a=0.08, w=.375),
  type='l', xlab='Time (s)', ylab=expression(italic(G)(italic(t))),
  main='', ylim=c(0, 1))

```



```

function F = ratcliff_cdf(t, v, a, w, eta2, sigma2, err)
%ratcliff_cdf: calculate CDF of FPT in a Ratcliff DDM to the lower barrier
% v is mean drift rate
% a is barrier separation
% w is relative starting point
% eta2 is drift rate variance;
% sigma2 is diffusion constant (usually 0.01)
% err is error tolerance of the infinite series truncation

F = zeros(1, length(t));
if(nargin < 7); err = sqrt(eps); end

if(any(t>0))
    sigma = sqrt(sigma2);
    F(t>0) = ratcliff_cdf1(t(t>0), v/sigma, a/sigma, w, eta2/sigma2, err);
end
return

function F = ratcliff_cdf1(t, v, a, w, eta2, err)

F = zeros(1, length(t));
sqt = sqrt(t);
denomMR = sqt.*sqrt(1+t*eta2);

j = 0;
while true %loop through pairs of even and odd j

    %even j
    rj = j*a + a*w;
    S1 = normpdf(rj./sqt) .* (M((rj - t*v + t*eta2*(rj + a*w)) ./ denomMR) + ...
        M((rj + t*v + t*eta2*(rj - a*w)) ./ denomMR));

    if(all(abs(S1) < err)); break; end
    j = j + 1;

    %odd j
    rj = j*a + a*(1-w);
    S2 = normpdf(rj./sqt) .* (M((rj - t*v + t*eta2*(rj + a*w)) ./ denomMR) + ...
        M((rj + t*v + t*eta2*(rj - a*w)) ./ denomMR));
    F = F + S1 - S2;

    if(all(abs(S2) < err)); break; end
    j = j + 1;
end
F = F .* exp((-t*v^2-2*v*a*w+eta2*a^2*w^2) ./ (2+2*t*eta2)); %prefactor
return

%calculate Mill's ratio
function M = M(x)
M = erfcx(x/sqrt(2)) / sqrt(2) * sqrt(pi);
return

```