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Non-planar ABJ theory and parity

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ABSTRACT

While the ABJ Chern–Simons-matter theory and its string theory dual manifestly lack parity invariance, no sign of parity violation has so far been observed on the weak coupling spin chain side. In particular, the planar two-loop dilatation generator of ABJ theory is parity invariant. In this Letter we derive the non-planar part of the two-loop dilatation generator of ABJ theory in its $SU(2)\times SU(2)$ sub-sector. Applying the dilatation generator to short operators, we explicitly demonstrate that, for operators carrying excitations on both spin chains, the non-planar part breaks parity invariance. For operators with only one type of excitation, however, parity remains conserved at the non-planar level. We furthermore observe that, as for ABJM theory, the degeneracy between planar parity pairs is lifted when non-planar corrections are taken into account.

1. Introduction

The concept of spin chain parity [1] played a crucial role in the discovery of higher loop integrability of the planar spectral problem of $\mathcal{N}=4$ SYM [2]. For a spin chain state the parity operation simply inverts the order of spins at the sites of the chain. In the field theory language the operation correspondingly inverts the order of fields inside a single trace operator or equivalently complex conjugates the gauge group generators. $\mathcal{N}=4$ SYM theory is parity invariant. In particular, the theory's dilatation generator commutes with parity. Integrability of the planar spectral problem at one loop order, discovered first in [3], implies the existence of a tower of higher conserved charges. The first of these, while commuting with the dilatation generator, anti-commutes with parity. As a consequence one finds in the planar spectrum pairs of operators with opposite parity but the same conformal dimension, denoted as planar parity pairs. The fact that these planar parity pairs survived higher loop corrections constituted the seed for the unveiling of higher loop integrability [2,4]. When non-planar corrections were taken into account, parity was still a good quantum number but the degeneracies between planar parity pairs disappeared [2]. While not disproving integrability this shows that the standard construction of conserved charges does not work any more.

The discovery of a novel AdS_4/CFT_3 correspondence [5,6] has provided us with the possibility of studying the effects of parity violation in a supersymmetric gauge theory and its dual string theory. A supersymmetric $\mathcal{N}=6$ Chern–Simons-matter theory with gauge group $U(M)_k\times \overline{U(N)}_{-k}$, where k denotes the Chern–Simons level, has been found to be dual to type IIA string theory on $AdS_4\times CP^3$ with a background NS B-field B_2 having non-trivial holonomy on $CP^1\subset CP^3$. More precisely[1]

$$\frac{1}{2\pi}\int\limits_{CP^1\subset CP^3} B_2=\frac{M-N}{k}. \tag{1}$$

This B-field holonomy causes breaking of world-sheet parity for $M\neq N$ and results in a string background which breaks target-space parity [6]. Correspondingly, the dual field theory does not respect three-dimensional parity invariance. For $M=N$ the Chern–Simons-matter theory is known as ABJM theory whereas the general version is denoted as ABJ theory. Our aim is to investigate how the parity breaking on the field theory side manifests itself in the spin chain language. The first steps in this direction were taken in [7,8] where the two-loop planar dilatation generator of ABJ theory was derived, respectively in an $SU(4)$ sub-sector and for the full set of fields. However, rather surprisingly, in these studies no effects of parity violation were seen. In fact the planar two-loop dilatation generator of ABJ theory differs from that of ABJM theory [9–11] only by an overall pre-factor. This raises the question of whether the parity symmetry of the

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[1] Here we have assumed that $M\geqslant N$. Quantum consistency of the theory requires in addition that $M-N\leqslant k$ [6].

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spin chain has a deeper significance, or is simply an accidental symmetry of the two-loop planar approximation. In the present Letter we will derive the two-loop non-planar dilatation generator of ABJ theory in a $SU(2) \times SU(2) \subset SU(4)$ sub-sector and explicitly demonstrate parity-breaking effects.

We start by, in Section 2, briefly describing ABJ theory and subsequently proceed to derive its full (planar plus non-planar) two-loop dilatation generator in the $SU(2) \times SU(2)$ sector in Section 3. As the derivation follows closely that of ABJM theory [12] we shall be very brief. In Section 4 we explicitly apply the dilatation generator to a series of short operators and determine their spectrum. In particular, we show that the non-planar part of the dilatation generator does *not* conserve parity. In addition, we observe a lifting of all planar degeneracies. Finally, Section 5 contains our conclusion.

2. ABJ theory

Our notation will follow that of Refs. [10,13]. ABJ theory [6] (see also [14] for a discussion at the classical level) is a three-dimensional $\mathcal{N}=6$ superconformal Chern–Simons-matter theory with gauge group $U(M)_k \times \overline{U(N)}_{-k}$ and R-symmetry group $SU(4)$. The parameter k denotes the Chern–Simons level. The fields of ABJ theory consist of gauge fields A_m and $\bar{A}_m$, complex scalars Y^I and Majorana spinors Ψ_I, $I \in \{1,\dots,4\}$. The two gauge fields A_m and $\bar{A}_m$ belong to the adjoint representation of $U(M)$ and $\overline{U(N)}$ respectively. For $N=M$, ABJ theory reduces to ABJM theory. The scalars Y^I and the spinors Ψ_I are bi-fundamental and transform in the $M \times \bar{N}$ representation of the gauge group and in the fundamental and anti-fundamental representation of $SU(4)$ respectively. For our purposes it proves convenient to write the scalars and spinors explicitly in terms of their $SU(2)$ component fields, i.e. [13]

$$Y^I = \{Z^A, W^{\dagger A}\}, \qquad Y_I^\dagger = \{Z_A^\dagger, W_A\}, \qquad \Psi_I = \{\epsilon_{AB}\xi^B e^{i\pi/4}, \epsilon_{AB}\omega^{\dagger B} e^{-i\pi/4}\}, \qquad \Psi^{I\dagger} = \{-\epsilon^{AB}\xi_B^\dagger e^{-i\pi/4}, -\epsilon^{AB}\omega_B e^{i\pi/4}\},$$

where now $A, B \in \{1,2\}$. Expressed in terms of these fields the action reads

$$S = \int d^3x \left[\frac{k}{4\pi}\epsilon^{mnp}\,\mathrm{Tr}\left(A_m\partial_n A_p + \frac{2i}{3}A_m A_n A_p\right) - \frac{k}{4\pi}\epsilon^{mnp}\,\mathrm{Tr}\left(\bar{A}_m\partial_n \bar{A}_p + \frac{2i}{3}\bar{A}_m \bar{A}_n \bar{A}_p\right) \right.$$
$$\left. - \mathrm{Tr}(\mathcal{D}_m Z)^\dagger \mathcal{D}^m Z - \mathrm{Tr}(\mathcal{D}_m W)^\dagger \mathcal{D}^m W + i\,\mathrm{Tr}\,\xi^\dagger \not{\mathcal{D}}\xi + i\,\mathrm{Tr}\,\omega^\dagger \not{\mathcal{D}}\omega - V_D^{\text{ferm}} - V_D^{\text{bos}} - V_F^{\text{ferm}} - V_F^{\text{bos}} \right].$$

Here the covariant derivatives are defined as

$$\mathcal{D}_m Z^A = \partial_m Z^A + iA_m Z^A - iZ^A \bar{A}_m, \qquad \mathcal{D}_m W_A = \partial_m W_A + i\bar{A}_m W_A - iW_A A_m, \tag{2}$$

and similarly for $\mathcal{D}_m\xi^B$ and $\mathcal{D}_m\omega_B$. The decomposition of the scalars and fermions into their $SU(2)$ components has allowed us to split the bosonic as well as the fermionic potential into D-terms and F-terms. The precise form of these can be found in [12]. The theory has two 't Hooft parameters

$$\lambda = \frac{4\pi N}{k}, \qquad \hat{\lambda} = \frac{4\pi M}{k}, \tag{3}$$

and one can consider the double 't Hooft limit

$$N, M \to \infty, \qquad k \to \infty, \qquad \lambda, \hat{\lambda} \text{ fixed.} \tag{4}$$

Furthermore, the theory has a multiple expansion in λ, $\hat{\lambda}$, $\frac{1}{N}$ and $\frac{1}{M}$. The action of three-dimensional parity flips the levels of the Chern–Simons terms, which produces a different theory if $M \neq N$. Thus the ABJ model is not parity invariant.

In this Letter we will be interested in studying non-planar corrections (i.e. $\frac{1}{N}$ and $\frac{1}{M}$ corrections) for anomalous dimensions at the leading two-loop level. We shall restrict ourselves to considering scalar operators belonging to a $SU(2) \times SU(2)$ sub-sector i.e. operators of the following type

$$\mathcal{O} = \mathrm{Tr}\big(Z^{A_1} W_{B_1} \cdots Z^{A_L} W_{B_L}\big), \tag{5}$$

where $A_i, B_i \in \{1,2\}$, and their multi-trace generalizations. A central object in our analysis will be the parity operation which acts on an operator by inverting the order of the fields inside each of its traces, i.e.[2]

$$P: \mathrm{Tr}\big(Z^{A_1} W_{B_1} \cdots Z^{A_L} W_{B_L}\big) \longrightarrow \mathrm{Tr}\big(W_{B_L} Z^{A_L} \cdots W_{B_1} Z^{A_1}\big). \tag{6}$$

Strictly speaking the parity operation (which would be a true symmetry in ABJM theory) involves in addition a complex conjugation of the fields [7] but as complex conjugating the fields inside an operator does not change its anomalous dimension the present definition suffices for our purposes.

3. The derivation of the full dilatation generator

The derivation of the full two-loop dilatation generator of ABJ theory is slightly lengthy but follows closely the one for ABJM theory [12]. The contractions one has to do are the same as before, only now one has to carefully keep track of whether a given contraction gives a factor of N or a factor of M. The Feynman diagrams which contribute at two-loop order consist of the ones depicted in Fig. 1 plus 14 self-energy diagrams. All diagrams of course come in planar as well as non-planar versions. In order to handle most easily the combinatorics of planar as well as non-planar diagrams it is again convenient to make use of the method of effective vertices [15]. An effective vertex is a space–time independent vertex which, when contracted with a given operator of the type (5) gives the combinatorial factor associated with a particular Feynman integral times the value of the integral. If things work as in $\mathcal{N}=4$ SYM and as in ABJM

[2] We notice that it is not possible to define in a natural and simple way a parity operation which acts only on Z or W fields.

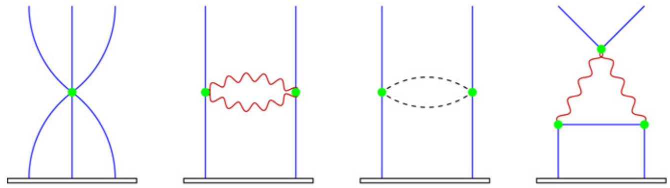

Fig. 1. The four types of two-loop diagrams contributing to anomalous dimensions. For operators in the $SU(2) \times SU(2)$ sector diagrams in class (d) do not contribute.

theory [12] the contribution from the bosonic D-terms should cancel against contributions from gluon exchange, fermion exchange and self-interactions to all orders in the genus expansion and this is indeed what happens. To prove this we first calculate the effective vertices corresponding to the four diagrams in Fig. 1. We notice, however, that for operators belonging to the $SU(2) \times SU(2)$ sector there are no contributions from Fig. 1d. Adding the contributions from the bosonic potential, gluon exchange and fermion exchange we find

$$\left(V^{\text{bos}}\right)^{\text{eff}} + \left(V^{\text{ferm}}\right)^{\text{eff}} + \left(V^{\text{gluon}}\right)^{\text{eff}} = \left(V_F^{\text{bos}}\right)^{\text{eff}} + V + \text{const}{:}\left\{\operatorname{Tr}\left(Z_C^{\dagger} Z^C\right) + \operatorname{Tr}\left(W_C W^{\dagger C}\right)\right\}{:}, \tag{7}$$

where

$$\text{const} = -\frac{1}{8}\left(\lambda^2 + \hat{\lambda}^2\right) - \frac{1}{2}\lambda\hat{\lambda} + \frac{5}{24}\frac{\lambda^2}{N^2} + \frac{5}{24}\frac{\hat{\lambda}^2}{M^2} + \frac{1}{3}\frac{\lambda}{N}\frac{\hat{\lambda}}{M}, \tag{8}$$

and where :: means that self-contractions should be excluded. The quantity V is a vertex which can be shown to give a vanishing contribution when applied to any operator in the $SU(2) \times SU(2)$ sector. Furthermore, the last term in Eq. (7) has exactly the form expected for self-energies and one can show that it precisely cancels the contribution from these. To do so one has to check the cancellation of both the planar and the non-planar part of the constant appearing in Eq. (8). The planar part of the analysis can be carried out with the aid of Ref. [7]. The non-planar part, however, requires a careful analysis of the non-planar versions of the 14 self-energy diagrams.

Collecting everything, we thus verify that the full two-loop dilatation generator is indeed given only by the F-terms in the bosonic potential, i.e.

$$D = \left(V_F^{\text{bos}}\right)^{\text{eff}} = -\frac{\lambda}{N}\frac{\hat{\lambda}}{M}{:}\operatorname{Tr}\left[W^{\dagger A} Z_B^{\dagger} W^{\dagger C} W_A Z^B W_C - W^{\dagger A} Z_B^{\dagger} W^{\dagger C} W_C Z^B W_A + Z_A^{\dagger} W^{\dagger B} Z_C^{\dagger} Z^A W_B Z^C - Z_A^{\dagger} W^{\dagger B} Z_C^{\dagger} Z^C W_B Z^A\right]{:}. \tag{9}$$

It is easy to see that the dilatation generator vanishes when acting on an operator consisting of only two of the four fields from the $SU(2) \times SU(2)$ sector. Accordingly we will denote two of the fields, say Z_1 and W_1, as background fields and Z_2 and W_2 as excitations. It is likewise easy to see that operators with only one type of excitation, say W_2's, form a closed set under dilatations. For operators with only W_2-excitations the dilatation generator consists of four terms whereas in the case with two different types of excitations it has 16 terms. In both cases D is easily seen to reduce to the one of [9,10] in the planar limit

$$D_{\text{planar}} \equiv \lambda\hat{\lambda} D_0 = \lambda\hat{\lambda} \sum_{k=1}^{2L} (1 - P_{k,k+2}), \tag{10}$$

where $P_{k,k+2}$ denotes the permutation between sites k and $k+2$ and $2L$ denotes the total number of fields inside an operator. It differs from the planar dilatation generator of ABJM theory only by having the pre-factor $\lambda\hat{\lambda}$ instead of λ^2. As explained in [9,10] this is the Hamiltonian of two alternating $SU(2)$ Heisenberg spin chains, coupled via a momentum condition. As mentioned earlier, integrability implies that there exists a tower of charges which all commute and which commute with the Hamiltonian. In particular, there exists one such charge Q_3 which anti-commutes with parity. In addition, the planar dilatation generator itself commutes with parity, i.e.

$$[D_{\text{planar}}, Q_3] = [D_{\text{planar}}, P] = \{Q_3, P\} = 0. \tag{11}$$

As a consequence, the spectrum of the planar theory has degenerate parity pairs, i.e. pairs of operators with identical anomalous dimension but opposite parity. In Ref. [12] it was shown that for ABJM theory at the non-planar level the two-loop dilatation generator still commutes with parity but the degeneracies between parity pairs are lifted. This hinted towards the absence of higher conserved charges, at least in a standard form. Below we will analyse the situation for ABJ theory and find that again the planar degeneracies disappear but in addition the non-planar two-loop dilatation generator does *not* any longer commute with parity.

When acting with the dilatation generator on a given operator we have to perform three contractions as dictated by the three Hermitian conjugate fields. It is easy to see that by acting with the dilatation generator one can change the number of traces in a given operator by at most two. More precisely, the two-loop dilatation generator has the expansion

$$D = \lambda\hat{\lambda}\left\{D_0 + \frac{1}{\mathcal{M}}(D_+ + D_-) + \frac{1}{\mathcal{M}^2}(D_{00} + D_{++} + D_{--})\right\}. \tag{12}$$

Here D_+ and D_{++} increase the number of traces by one and two respectively and D_- and D_{--} decrease the number of traces by one and two. Finally, D_0 does not change the number of traces and D_{00} first adds one trace and subsequently removes one or vice versa. The quantity $\frac{1}{\mathcal{M}}$ stands for $\frac{1}{N}$ or $\frac{1}{M}$ and $\frac{1}{\mathcal{M}^2}$ stands for $\frac{1}{N^2}$, $\frac{1}{M^2}$ or $\frac{1}{MN}$.

Even for short operators it is in practice hard to diagonalise the full dilatation generator exactly. But one can relatively easily diagonalise the planar dilatation generator, either by brute force or by means of the Bethe equations. Subsequently the non-planar terms can be treated as perturbations and the energy corrections found approximately using quantum mechanical perturbation theory [16]. Notice that while energy corrections are generically of order $\frac{1}{\mathcal{M}^2}$, degeneracies in the planar spectrum will lead to energy corrections of order $\frac{1}{\mathcal{M}}$. (For details see [12].)

4. Short operators

In this section we determine non-planar corrections to the anomalous dimensions of a number of short operators. This is done by explicitly computing and diagonalising the planar mixing matrix (aided by `GPL Maxima` as well as `Mathematica`) and subsequently determining the non-planar corrections by quantum mechanical perturbation theory.

4.1. Operators with excitations on the same chain

In this sector, the simplest set of operators for which one observes degenerate parity pairs as well as non-trivial mixing between operators with one, two and three traces consists of operators of length 14 with three excitations. There are in total 17 such non-protected operators. Among the non-protected operators there are only eight which are not descendants. Their explicit form can be found in Ref. [12]. The planar anomalous dimensions (in units of $\lambda\hat{\lambda}$), trace structure and parity for these eight operators, denoted as $\mathcal{O}_1, \ldots, \mathcal{O}_8$, are

Eigenvector	Eigenvalue	Trace structure	Parity
$\mathcal{O}_1$	5	(14)	$-$
$\mathcal{O}_2$	6	(2)(12)	$-$
$\mathcal{O}_3$	5	(14)	$+$
$\mathcal{O}_4$	$5+\sqrt{5}$	(2)(12)	$+$
$\mathcal{O}_5$	$5-\sqrt{5}$	(2)(12)	$+$
$\mathcal{O}_6$	4	(4)(10)	$+$
$\mathcal{O}_7$	4	(2)(2)(10)	$+$
$\mathcal{O}_8$	6	(2)(4)(8)	$+$

We have one pair of degenerate single trace operators with opposite parity, namely the operators $\mathcal{O}_1$ and $\mathcal{O}_3$.[3]

Expressing the dilatation generator in the basis above and taking into account all non-planar corrections we get (in units of $\lambda\hat{\lambda}$)[4]

$$\left(\begin{array}{cc|cccccc}
5+\frac{15}{MN} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{N}+\frac{3}{M} & 6+\frac{24}{MN} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 5+\frac{35}{MN} & 0 & 0 & -\frac{4}{N}-\frac{4}{M} & -\frac{4}{MN} & -\frac{2}{MN} \\
0 & 0 & -\frac{\sqrt{5}/2}{M}-\frac{\sqrt{5}/2}{N} & \sqrt{5}+5+\frac{(5\sqrt{5}+35)}{MN} & \frac{3\sqrt{5}-5}{MN} & \frac{1}{MN} & 0 & \frac{1}{M}+\frac{1}{N} \\
0 & 0 & -\frac{\sqrt{5}/2}{M}-\frac{\sqrt{5}/2}{N} & -\frac{5+3\sqrt{5}}{MN} & 5-\sqrt{5}-\frac{5\sqrt{5}-35}{MN} & -\frac{1}{MN} & 0 & -\frac{1}{M}-\frac{1}{N} \\
0 & 0 & -\frac{10}{N}-\frac{10}{M} & \frac{4\sqrt{5}+20}{MN} & -\frac{20-4\sqrt{5}}{MN} & 4+\frac{28}{MN} & 0 & 0 \\
0 & 0 & -\frac{10}{MN} & \frac{2\sqrt{5}+10}{N}+\frac{2\sqrt{5}+10}{M} & \frac{2\sqrt{5}-10}{N}+\frac{2\sqrt{5}-10}{M} & 0 & 4+\frac{32}{MN} & -\frac{2}{MN} \\
0 & 0 & -\frac{10}{MN} & \frac{12\sqrt{5}+20}{N}+\frac{12\sqrt{5}+20}{M} & \frac{12\sqrt{5}-20}{N}+\frac{12\sqrt{5}-20}{M} & \frac{4}{N}+\frac{4}{M} & -\frac{8}{MN} & 6+\frac{40}{MN}
\end{array}\right).$$

This mixing matrix of course reduces to that of ABJM theory for $N=M$ as it should, cf. [12]. We notice that for this type of operators the positive and negative parity states still decouple, i.e. parity is preserved. The states $\mathcal{O}_1$ and $\mathcal{O}_2$ are exact eigenstates of the full dilatation generator with non-planar corrections equal to

$$\delta E_1 = \frac{15}{NM}, \qquad \delta E_2 = \frac{24}{NM}. \tag{13}$$

For the remaining operators we observe that all matrix elements between degenerate states vanish. Thus the leading non-planar corrections to the anomalous dimensions can be found using second order non-degenerate perturbation theory. The results read

$$\begin{aligned}
&\delta E_3 = \frac{40}{M^2}+\frac{40}{N^2}+\frac{115}{MN}, \qquad \delta E_4 = 4(5+2\sqrt{5})\left(\frac{1}{N^2}+\frac{1}{M^2}\right)+\frac{3(25+7\sqrt{5})}{MN}, \\
&\delta E_5 = 4(5-2\sqrt{5})\left(\frac{1}{N^2}+\frac{1}{M^2}\right)+\frac{3(25-7\sqrt{5})}{MN}, \qquad \delta E_6 = -\frac{40}{N^2}-\frac{40}{M^2}-\frac{52}{MN}, \\
&\delta E_7 = \frac{32}{MN}, \qquad \delta E_8 = -40\left(\frac{1}{N^2}+\frac{1}{M^2}\right)-\frac{40}{MN}.
\end{aligned} \tag{14}$$

We observe that all degeneracies found at the planar level get lifted when non-planar corrections are taken into account, for all values of M and N. This in particular holds for the degeneracies between the members of the planar parity pair $(\mathcal{O}_1, \mathcal{O}_3)$. We have considered a number of different types of states with only one type of excitation and have found that the same pattern persists in all cases. In fact, one can explicitly show that the matrix elements between n and $(n+1)$–trace states of the normal ordered operator in Eq. (9) (i.e. D without its pre-factor) can only depend on M and N through the combination $M+N$. Thus one cannot have parity breaking.

4.2. Operators with excitations on both chains

The simplest multiplet of operators which have non-planar energy corrections are operators of length six with two excitations. There are in total three such non-protected highest weight states. These read

[3] We also observe a degeneracy between the negative parity double trace state $\mathcal{O}_2$ and the positive parity triple trace state $\mathcal{O}_8$ as well as a degeneracy between the double trace state $\mathcal{O}_6$ and the triple trace state $\mathcal{O}_7$ both of positive parity. However, states with different numbers of traces cannot be connected via the conserved charge Q_3.

[4] Notice that by construction the mixing matrix is not Hermitian but related to its Hermitian conjugate by a similarity transformation [16,17].

$$
\begin{aligned}
\mathcal{O}_1 &= \mathrm{Tr}(Z_1W_1Z_1W_2Z_2W_1) + \mathrm{Tr}(Z_1W_1Z_1W_1Z_2W_2) - 2\,\mathrm{Tr}(Z_1W_1Z_2W_1Z_1W_2),\\
\mathcal{O}_2 &= \mathrm{Tr}(Z_1W_1Z_1W_2Z_2W_1) - \mathrm{Tr}(Z_1W_1Z_1W_1Z_2W_2),\\
\mathcal{O}_3 &= \mathrm{Tr}(Z_1W_1)\,\mathrm{Tr}(Z_1W_1Z_2W_2) - \mathrm{Tr}(Z_1W_1)\,\mathrm{Tr}(Z_1W_2Z_2W_1).
\end{aligned} \tag{15}
$$

Their associated planar anomalous dimension (in units of $\lambda\hat{\lambda}$), parity and trace structure are

Eigenvector	Eigenvalue	Trace structure	Parity
$\mathcal{O}_1$	6	(6)	+
$\mathcal{O}_2$	6	(6)	−
$\mathcal{O}_3$	8	(2)(4)	−

Already in this simple case we have one pair of degenerate states with opposite parity, namely $\mathcal{O}_1$ and $\mathcal{O}_2$. Expressing the dilatation generator in this basis and taking into account all non-planar corrections we get (in units of $\lambda\hat{\lambda}$)

$$
\left(\begin{array}{c|cc}
6 & 0 & \frac{1}{M}-\frac{1}{N}\\
\hline
0 & 6-\frac{12}{MN} & -\frac{3}{M}-\frac{3}{N}\\
\frac{6}{M}-\frac{6}{N} & -\frac{6}{M}-\frac{6}{N} & 8-\frac{8}{MN}
\end{array}\right).
$$

We observe that in this case the dilatation generator does mix states with different parity. In other words, the non-planar dilatation generator does *not* commute with P. Calculating the energies by second order quantum mechanical perturbation theory we find

$$
\delta E_1 = -\frac{3}{N^2} - \frac{3}{M^2} + \frac{6}{MN}, \qquad \delta E_2 = -\frac{9}{M^2} - \frac{9}{N^2} - \frac{30}{MN}, \qquad \delta E_3 = \frac{4}{M^2} + \frac{4}{N^2} + \frac{4}{MN}. \tag{16}
$$

In particular, we see that the planar degeneracy is lifted.

Let us analyse a slightly larger multiplet of operators with two excitations of different types that exhibit some more of the above mentioned non-trivial features of the topological expansion: Operators of length eight with one excitation of each type. There are in total 7 such non-protected operators. Their explicit form can be found in Ref. [12] and the planar anomalous dimensions (in units of $\lambda\hat{\lambda}$), trace structure and parity of these operators, denoted as $\mathcal{O}_1, \ldots, \mathcal{O}_7$, are

Eigenvector	Eigenvalue	Trace structure	Parity
$\mathcal{O}_1$	8	(8)	−
$\mathcal{O}_2$	4	(8)	−
$\mathcal{O}_3$	8	(4)(4)	−
$\mathcal{O}_4$	6	(2)(6)	−
$\mathcal{O}_5$	8	(2)(2)(4)	−
$\mathcal{O}_6$	4	(8)	+
$\mathcal{O}_7$	6	(2)(6)	+

Notice that we have two pairs of degenerate operators with opposite parity, namely the single trace operators $\mathcal{O}_2$ and $\mathcal{O}_6$ and the double trace operators $\mathcal{O}_4$ and $\mathcal{O}_7$.[5]

Expressing the dilatation generator in the basis given above and taking into account all non-planar corrections we get (in units of $\lambda\hat{\lambda}$)

$$
\left(\begin{array}{ccccc|cc}
8 & \frac{8}{MN} & \frac{8}{N}+\frac{8}{M} & \frac{2}{N}+\frac{2}{M} & -\frac{8}{MN} & 0 & \frac{2}{M}-\frac{2}{N}\\
\frac{8}{MN} & 4-\frac{12}{MN} & 0 & -\frac{1}{N}-\frac{1}{M} & -\frac{4}{MN} & 0 & \frac{1}{N}-\frac{1}{M}\\
\frac{8}{N}+\frac{8}{M} & -\frac{4}{N}-\frac{4}{M} & 8 & 0 & 0 & \frac{4}{M}-\frac{4}{N} & 0\\
0 & -\frac{8}{N}-\frac{8}{M} & -\frac{8}{MN} & 6-\frac{8}{MN} & -\frac{6}{N}-\frac{6}{M} & \frac{4}{M}-\frac{4}{N} & 0\\
0 & \frac{8}{MN} & 0 & -\frac{6}{N}-\frac{6}{M} & 8-\frac{8}{MN} & 0 & \frac{6}{N}-\frac{6}{M}\\
\hline
0 & 0 & 0 & \frac{1}{M}-\frac{1}{N} & 0 & 4+\frac{4}{MN} & \frac{1}{N}+\frac{1}{M}\\
0 & 0 & 0 & 0 & \frac{2}{N}-\frac{2}{M} & \frac{4}{N}+\frac{4}{M} & 6+\frac{8}{MN}
\end{array}\right).
$$

This mixing matrix of course reduces to that of ABJM theory for $N = M$ as it should, cf. [12]. We observe again that the dilatation generator does mix states with different parity. To find the corrections to the eigenvalues we use perturbation theory as described in Section 3. First, we notice that most matrix elements between degenerate states vanish. The only exception are the matrix elements between the states $\mathcal{O}_1$ and $\mathcal{O}_3$. To find the non-planar correction to the energy of these states we diagonalise the Hamiltonian in the corresponding subspace and find

$$
\delta E_{1,3} = \mp\left(\frac{8}{N} + \frac{8}{M}\right). \tag{17}
$$

For the remaining operators the leading non-planar corrections to the energy can be found using second order non-degenerate perturbation theory. The results read

$$
\begin{aligned}
&\delta E_2 = -\frac{20}{NM} - \frac{4}{N^2} - \frac{4}{M^2}, \qquad \delta E_4 = -\frac{40}{NM} - \frac{12}{N^2} - \frac{12}{M^2},\\
&\delta E_5 = \frac{16}{NM} + \frac{24}{N^2} + \frac{24}{M^2}, \qquad \delta E_6 = \frac{4}{MN} - \frac{4}{N^2} - \frac{4}{M^2}, \qquad \delta E_7 = \frac{24}{MN} - \frac{4}{N^2} - \frac{4}{M^2}.
\end{aligned}
$$

[5] The double trace operators $\mathcal{O}_4$ and $\mathcal{O}_7$ can be related via Q_3 when letting Q_3 act only on the longer of the two constituent traces of the operators.

We again notice that all degeneracies observed at the planar level get lifted when non-planar corrections are taken into account, for all values of M and N. This in particular holds for the degeneracies between the members of the two parity pairs. We have examined a number of operators with excitations of two different types and found that the same pattern persists in all cases. A closer scrutiny of the action of the dilatation generator reveals that the asymmetry between M and N originates from the situation where the operator separates two neighbouring excitations, a situation which one does not encounter when the two excitations are on the same chain. Let us note that the characteristic polynomial of the anomalous dimension matrices will always be even in $M - N$. This implies that the eigenvalues will generically be even under the interchange of M and N (as is the case above). A possible exception might arise in cases where nonzero matrix elements appear between planar degenerate states which have opposite parity and differ in trace number by one (notice that the requirement of different trace structure prevents this complication from arising for planar parity pairs). Although mixing of the above type does occur, we did not observe any asymmetry in the eigenvalues for the explicit cases we examined.

5. Conclusion

We have derived and analysed the non-planar corrections to the two-loop dilatation generator of ABJ theory in the $SU(2) \times SU(2)$ sub-sector. Our analysis shows that these corrections mix states with positive and negative parity, i.e.

$$\left[D^{\mathrm{ABJ}}_{\text{non-planar}}, P\right] \neq 0. \tag{18}$$

More precisely, the value of the commutator is proportional to $M - N$. This is in contrast to earlier studies of the *planar* two-loop dilatation generator which did not reveal any sign of parity breaking [7,8]. Furthermore, whereas the planar dilatation generator could be proved to be integrable, we do not see any indication of this being the case for the non-planar one, since none of the planar degeneracies between parity pairs survive the inclusion of non-planar corrections. It is an interesting question whether the planar dilatation generator remains integrable and parity invariant when higher loop corrections are taken into account. In this connection it is worth mentioning that parity breaking does not prevent integrability [7,8]. At planar level, one could try to address the question of parity breaking at higher-loop order from the string theory side by calculating a transition amplitude between two string states of different parity living in an instanton background of the ABJ theory dual. We note that an interesting effect of parity breaking in the non-interacting string theory has been observed in [18].

One could also try to match the results of the present calculation to the behaviour of the dual string theory by calculating the semi-classical amplitude for non-parity-conserving splitting of a one-string state into a two-string state in the spirit of [19,20]. Of course, this calculation would at best allow us to obtain qualitative agreement between non-planar gauge theory and interacting string theory. How to achieve quantitative agreement remains a challenge.

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