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# THE INTEGRATION ORDER OF VECTOR AUTOREGRESSIVE PROCESSES

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Abstract. We show that the order of integration of a vector autoregressive process is equal to the difference between the multiplicity of the unit root in the characteristic equation and the multiplicity of the unit root in the adjoint matrix polynomial. The equivalence with the standard $I(1)$ and $I(2)$ conditions (Johansen, 1996) is proved and polynomial cointegration discussed in the general setup.

## 1. Introduction

An autoregressive process is integrated of order $d$, if its characteristic equation has $d$ roots at $z = 1$ and the remaining lie outside the unit circle. This is not true in the multivariate case, because the order of integration of a vector autoregressive processes is not established by the multiplicity of the unit root in the characteristic equation. For this reason, some extra conditions are needed in order to write down the moving average representation. Johansen (1988, 1992) imposes necessary and sufficient conditions on the parameters of the autoregressive process and derives the corresponding moving average representation for $I(1)$ and $I(2)$ processes. His work is related to Engle and Granger (1987), who start from the moving average representation of an $I(1)$ process which exhibits cointegration and derive the corresponding error correction model; unfortunately the proof of the Granger Representation Theorem is not correct (see Johansen (2005a) for a counterexample to Lemma 1). Other proofs of the same theorem are based on the Smith

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form of a matrix polynomial (see Engle and Yoo (1991), Ahn and Reinsel (1990) and Haldrup and Salomon (1998)) and on the Jordan representation of the companion form (see Archontakis (1998) and Bauer and Wagner (2003)). Other relevant papers in this area are Gregoire and Laroque (1993) and Gregoire (1999), who discuss polynomial cointegration in a very general setup and Neusser (2000), who points out some interesting algebraic properties of the $I(1)$ model. An attempt to characterize explicitly the polynomial cointegration properties of an $I(d)$ process from its autoregressive representation is la Cour (1998). See Johansen (2005a) for an exhaustive survey of the mathematical results concerning the representation theory and Johansen (2005b) for an application of similar ideas to fractional integration and cofractionality.

In this paper we study the $I(d)$ multivariate case and show that one can determine the order of integration of a vector autoregressive process as the difference between the multiplicity of the unit root in the characteristic equation and the multiplicity of the unit root in the adjoint matrix polynomial. This result arises from observing that the reduced rank of the characteristic polynomial at $z = 1$ translates into a zero versus non zero statement about the adjoint matrix polynomial. This then allows to write the inverse in such a way that the order of the pole at the unit root becomes explicit, resembling what happens in the univariate case.

The paper is organized as follows: in section 2 we introduce the VAR(k) model and the standard definitions of integration and cointegration and in section 3 we prove the main Theorem on $I(d)$ processes. In section 4 we show the equivalence with the standard $I(1)$ and $I(2)$ conditions (Johansen, 1996) and in section 5 we discuss polynomial cointegration. The last section contains some concluding remarks.

## 2. VAR(k) model and standard definitions

Consider the $p-$dimensional autoregressive model

$$X_t = \Pi_1 X_{t-1} + \Pi_2 X_{t-2} + \cdots + \Pi_k X_{t-k} + \epsilon_t, \tag{2.1}$$

or $\Pi(L)X_t = \epsilon_t$ and $\epsilon_t$ is an i.i.d. process.

**Definition 2.1.** *The process $X_t = C(L)\epsilon_t$ is stationary if $C(z) = \sum_{i=0}^{\infty} C_i z^i$ converges for $|z| < 1 + \delta$ for some $\delta > 0$; it is $I(0)$ when it is stationary and $C(1) \neq 0$; it is $I(d)$, $d > 0$, if $\Delta^d X_t$ is $I(0)$.*

Cointegration and polynomial cointegration are defined as follows

**Definition 2.2.** *The $I(d)$ process $X_t$ is cointegrated if there exists $\beta$ such that $\beta' X_t$ is $I(b)$ with $b < d$. It is polynomially cointegrated if there exists $\beta_k$ for $k = 0, \cdots, d-1$, such that $\sum_{k=0}^{d-1} \beta_k' \Delta^k X_t$ is stationary.*

## 3. Poles, order of integration and multiplicities

The characteristic polynomial of (2.1) is the $p \times p$ matrix function

$$\Pi(z) = I_p - \sum_{i=1}^{k} \Pi_i z^i,$$

and the characteristic equation is defined as $|\Pi(z)| = 0$, where $|\Pi(z)| = det(\Pi(z))$ is a polynomial of degree $n \leq kp$, $|\Pi(z)| = \sum_{i=0}^{n} d_i z^i$. From $|\Pi(0)| = 1$ it follows that zero is not a root of the characteristic equation. Let $n_r$ be the number of distinct roots $z_i$, each with multiplicity $m_i$; the determinant can thus be written as

$$|\Pi(z)| = d_n \prod_{i=1}^{n_r} (z - z_i)^{m_i} = (z-1)^m g(z), \tag{3.1}$$

where $g(1) \neq 0$ and $1 \leq m \leq n$.

**Assumption 3.1.** *The only unstable root is at $z = 1$; that is $|\Pi(z)| = 0$ implies $z_i = 1$ or $|z_i| > 1$.*

Evaluating the characteristic polynomial at the roots of the characteristic equation we get reduced rank matrices; at the unit root we write $\Pi(1) = -\alpha\beta'$, where $\alpha$ and $\beta$ are $p \times r$ matrices of full rank $r < p$.

The inverse is defined as the adjoint matrix $\Pi_a(z) = Adj(\Pi(z))$ divided by the determinant

$$\Pi(z)^{-1} = \frac{\Pi_a(z)}{|\Pi(z)|}, \; z \neq \{1, \cdots, z_{n_r}\}. \tag{3.2}$$

Since $\Pi(z)$ has reduced rank at the roots of the characteristic equation, (3.2) is not defined at $z = \{1, \cdots, z_{n_r}\}$. These singularities are known to be poles but at the moment nothing can be said about their order.

**Proposition 3.2.** *If Assumption 3.1 holds, then $X_t$ is $I(d)$ if and only if $\Pi(z)^{-1}$ has a pole of order $d$ at $z = 1$.*

Proof. By definition $X_t$ is $I(d)$ if $\Delta^d X_t = C(L)\epsilon_t$ with $C(z) = \sum_{i=0}^{\infty} C_i z^i$ convergent for $|z| < 1 + \delta$ for some $\delta > 0$ and $C(1) \neq 0$; then $\Pi(z)^{-1} = C(z)/(1-z)^d$ has a pole of order $d$ at $z = 1$. ■

Now consider the numerator in (3.2) at $z = 1$: when we evaluate the adjoint matrix polynomial at the unit root we have that[1]

$$\Pi_a(1) = 0_p \text{ when } r < p - 1,$$

because the determinant of any $p-1 \times p-1$ minor extracted from $\Pi(1)$ is zero and

$$\Pi_a(1) \neq 0_p \text{ when } r = p - 1,$$

because there is at least one non singular $p - 1 \times p - 1$ minor in $\Pi(1)$.

It follows that $z-1$ can be factored out $a$ times from $\Pi_a(z)$, for some $a > 0$ when $r < p - 1$ and for $a = 0$ when $r = p - 1$; consequently we have that

$$\Pi_a(z) = (z - 1)^a H(z),$$

where $H(1) \neq 0_p$ and $a \geq 0$.

The only way of having a pole at $z = 1$ in (3.2) is if $\Pi_a(z)$ goes to zero at a slower rate than $|\Pi(z)|$. Equivalently, we could say that it must be the case that $a < m$.

This is exactly what Theorem 3.3 below makes precise.

**Theorem 3.3.** *If Assumption 3.1 holds, the order of integration of $X_t$ is equal to*

$$d = m - a,$$

*where $H(1) \neq 0_p$ in $\Pi_a(z) = (z - 1)^a H(z)$ and $m$ is the multiplicity of the unit root in the characteristic equation.*

Proof. Assume that $a = m - d$ and $H(1) \neq 0_p$; then

$$\Pi(z)^{-1} = \frac{\Pi_a(z)}{|\Pi(z)|} = \frac{H(z)}{(z-1)^d g(z)},$$

from which we see that $\Pi(z)^{-1}$ has a pole of order $d$ at $z = 1$; thus $X_t$ is $I(d)$ by Proposition 3.2. Assume now that $X_t$ is $I(d)$; by Proposition 3.2 it follows that $\Pi(z)^{-1}$ has a pole of order $d$ at $z = 1$. Then $(z - 1)^{(d-1)}\Pi(z)^{-1}$ is not defined at $z = 1$ and $G(z) = (z-1)^d\Pi(z)^{-1}$ is such that $G(1) \neq 0_p$. This defines $H(z) = \Pi_a(z)/(z - 1)^{(m-d)}$ which implies $a = m - d$ and $H(1) = g(1)G(1) \neq 0_p$. ■

Thus the order of integration of the process is simply equal to the multiplicity of the unit root in the characteristic equation minus the multiplicity of the unit root in the adjoint matrix polynomial. We know how to calculate the roots of a polynomial; then it is clear that we can

[1]We use $0_p$ for the $p \times p$ zero matrix and $I_p$ for the identity matrix of the same dimension. For non square matrices we write both the row and column indexes.

find $m$, the number of roots at $z = 1$ in (3.1) and $m_{ij}$, the number of roots at $z = 1$ in entry $i, j$ of $\Pi_a(z)$. Then $a = \min_{ij} m_{ij}$ and the process is integrated of order $d = m - a$.

**Example 1** (Johansen, 1992): *Consider the model*

$$-\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} X_t + \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 2+\delta \end{pmatrix} \Delta X_t + \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \Delta^2 X_t = \epsilon_t,$$

*with characteristic polynomial*

$$\Pi(z) = \begin{pmatrix} -1 & -2 + \frac{z}{2}(1-z) \\ -2 + \frac{z}{2}(1-z) & -3 + \delta(1-z) - z^2 \end{pmatrix},$$

*and characteristic equation*

$$|\Pi(z)| = -(1-z)(\delta + 1 - z + \frac{z^2}{4}(1-z)).$$

*Assumption 3.1 is satisfied if $\delta = 0$ or $\delta \geq 3$. When $\delta \geq 3$, $m = 1$ and $g(z) = \delta + 1 - z + \frac{z^2}{4}(1-z)$ is such that $g(1) = \delta$. Since $\Pi_a(1) \neq 0$ we have that $a = 0$ and $d = m = 1$. When $\delta = 0$, $m = 2$ and $g(z) = 1 + \frac{1}{4}z^2$ is such that $g(1) = \frac{5}{4}$; then $\Pi_a(1) \neq 0$ implies $d = 2$.*

**Example 2** (Paruolo, 1996): *Consider the model*

$$X_t = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 2 \end{pmatrix} X_{t-1} + \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -1 \end{pmatrix} X_{t-2} + \epsilon_t,$$

*with characteristic polynomial*

$$\Pi(z) = \begin{pmatrix} 1 & 0 & -\frac{z}{2}(1-z) \\ 0 & 1-z & 0 \\ -\frac{z}{2}(1-z) & 0 & (1-z)^2 \end{pmatrix},$$

*and characteristic equation*

$$|\Pi(z)| = (1-z)^3(1 - \frac{z^2}{4}).$$

*Then Assumption 3.1 is satisfied, $m = 3$ and $g(z) = 1 - \frac{z^2}{4}$ is such that $g(1) = \frac{3}{4}$; the adjoint matrix polynomial is*

$$\Pi_a(z) = \begin{pmatrix} (1-z)^3 & 0 & \frac{z}{2}(1-z)^2 \\ 0 & (1-z)^2(1-\frac{z^2}{4}) & 0 \\ \frac{z}{2}(1-z)^2 & 0 & 1-z \end{pmatrix},$$

*from which it is easily seen that $a = 1$ and thus that $d = m - a = 2$; thus the process is integrated of order 2.*

## 4. Equivalence with the standard $I(1)$ and $I(2)$ conditions

We want to prove the equivalence with the standard conditions in Johansen (1996) and derive the explicit expression of $H(1)$. We introduce the following notation: let $A_\perp$ be the orthogonal complement of an $m \times n$ matrix $A$ of rank $n < m$, let $\bar{A} = A(A'A)^{-1}$ and write the Taylor expansion of $\Pi(z)$ at $z = 1$ as

$$\Pi(z) = \Pi(1) + \dot{\Pi}(1)(z-1) + \frac{\ddot{\Pi}(1)}{2}(z-1)^2 + (z-1)^3\Pi_3(z).$$

The order of integration is established (Johansen, 1996) by some reduced and full rank conditions on specific matrices: $X_t$ is $I(1)$ if and only if $|\Pi(1)| = 0$ and $|\alpha'_\perp \dot{\Pi}(1)\beta_\perp| \neq 0$; similarly, the $I(2)$ condition states that the order of integration is two if and only if $|\Pi(1)| = 0$, $|\alpha'_\perp \dot{\Pi}(1)\beta_\perp| = 0$ and $|\alpha'_2\theta\beta_2| \neq 0$ where $\theta = \frac{\ddot{\Pi}(1)}{2} + \dot{\Pi}(1)\bar{\beta}\bar{\alpha}'\dot{\Pi}(1)$, $\alpha_2 = \alpha_\perp\xi_\perp$, $\beta_2 = \beta_\perp\eta_\perp$, $\alpha'_\perp\dot{\Pi}(1)\beta_\perp = \xi\eta'$ and $\xi, \eta$ are $p - r \times s$ matrices of full rank $s < p - r$. Using (3.1) and Theorem 3.3 we rewrite the identity $\Pi(z)\Pi_a(z) = \Pi_a(z)\Pi(z) = |\Pi(z)|I_p$ as

$$\Pi(z)H(z) = (z-1)^d g(z) I_p \tag{4.1}$$

and $H(z)\Pi(z) = (z-1)^d g(z) I_p$. At $z = 1$ they read $\alpha\beta' H(1) = H(1)\alpha\beta' = 0_p$ and mean that

$$H(1) = \beta_\perp \zeta_d \alpha'_\perp,$$

for some $p - r \times p - r$ matrix $\zeta_d$ of rank $0 < r_d \leq p - r$.

**Proposition 4.1** ($I(1)$ case)**.** *A necessary and sufficient condition for $|\alpha'_\perp \dot{\Pi}(1)\beta_\perp| \neq 0$ is that*

$$a = m - 1$$

*in $\Pi_a(z) = (z-1)^a H(z)$ and $H(1) \neq 0_p$. The explicit expression for $H(1)$ is*

$$H(1) = g(1)\beta_\perp(\alpha'_\perp\dot{\Pi}(1)\beta_\perp)^{-1}\alpha'_\perp.$$

Proof. Assume $a = m - 1$ so that $d = 1$; differentiate (4.1) at $z = 1$ to get $\dot{\Pi}(1)H(1) - \alpha\beta'\dot{H}(1) = g(1)I_p$ and thus

$$\alpha'_\perp\dot{\Pi}(1)\beta_\perp\zeta_1 = g(1)I_{p-r}.$$

Then $g(1) \neq 0$ implies $|\zeta_1| \neq 0$, $|\alpha'_\perp\dot{\Pi}(1)\beta_\perp| \neq 0$ and $\zeta_1 = g(1)(\alpha'_\perp\dot{\Pi}(1)\beta_\perp)^{-1}$, and thus the $I(1)$ condition is satisfied.

Assume now the $I(1)$ condition holds and suppose $d = m - a > 1$; differentiating (4.1) at $z = 1$ we get $\alpha'_\perp\dot{\Pi}(1)\beta_\perp\zeta_d = 0_{p-r}$; since $\zeta_d \neq 0_{p-r}$ this contradicts $|\alpha'_\perp\dot{\Pi}(1)\beta_\perp| \neq 0$ and implies $m - a = 1$. ■

In the next proposition we consider the $I(2)$ case:

**Proposition 4.2** ($I(2)$ case)**.** *A necessary and sufficient condition for $\alpha'_\perp \dot{\Pi}(1)\beta_\perp = \xi\eta'$ and $|\alpha'_2\theta\beta_2| \neq 0$, where $\xi$ and $\eta$ are $p-r\times s$ matrices of full rank $s < p-r$, $\theta = \frac{\ddot{\Pi}(1)}{2} + \dot{\Pi}(1)\bar{\beta}\bar{\alpha}'\dot{\Pi}(1)$, $\alpha_2 = \alpha_\perp\xi_\perp$ and $\beta_2 = \beta_\perp\eta_\perp$ is that*

$$a = m - 2$$

*in $\Pi_a(z) = (z-1)^a H(z)$ and $H(1) \neq 0_p$. The explicit expression for $H(1)$ is*

$$H(1) = g(1)\beta_2(\alpha'_2\theta\beta_2)^{-1}\alpha'_2.$$

Proof. Assume $a = m-2$ so that $d = 2$; the first derivative of $\Pi(z)H(z) = H(z)\Pi(z) = (z-1)^2 g(z) I_p$ implies $\alpha'_\perp\dot{\Pi}(1)\beta_\perp\zeta_2 = \zeta_2\alpha'_\perp\dot{\Pi}(1)\beta_\perp = 0_{p-r}$ and thus $|\zeta_2||\alpha'_\perp\dot{\Pi}(1)\beta_\perp| = 0$; if $|\alpha'_\perp\dot{\Pi}(1)\beta_\perp| \neq 0$ then $\zeta_2 = 0_{p-r}$ contradicts $H(1) \neq 0_p$; thus $|\alpha'_\perp\dot{\Pi}(1)\beta_\perp| = 0$, $\alpha'_\perp\dot{\Pi}(1)\beta_\perp = \xi\eta'$ where $\xi$ and $\eta$ are $p-r\times s$ matrices of full rank $s < p-r$ and $\zeta_2 = \eta_\perp\psi\xi'_\perp$ for some $p-r-s\times p-r-s$ matrix $\psi$ of rank $0 < t \leq p-r-s$; then $H(1)$ becomes

$$H(1) = \beta_\perp\eta_\perp\psi\xi'_\perp\alpha'_\perp = \beta_2\psi\alpha'_2.$$

Note that the first derivative of (4.1) provides the equality

$$\beta'\dot{H}(1) = \bar{\alpha}'\dot{\Pi}(1)H(1). \tag{4.2}$$

The the second derivative of (4.1) implies

$$\alpha'_\perp\ddot{\Pi}(1)H(1) + 2\alpha'_\perp\dot{\Pi}(1)\dot{H}(1) = 2g(1)\alpha'_\perp. \tag{4.3}$$

Using $I_p = \beta_\perp\bar{\beta}'_\perp + \bar{\beta}\beta'$ we see that $\alpha'_\perp\dot{\Pi}(1)\dot{H}(1) = \xi\eta'\bar{\beta}'_\perp\dot{H}(1) + \alpha'_\perp\dot{\Pi}(1)\bar{\beta}\beta'\dot{H}(1) = \xi\eta'\bar{\beta}'_\perp\dot{H}(1) + \alpha'_\perp\dot{\Pi}(1)\bar{\beta}\bar{\alpha}'\dot{\Pi}(1)H(1)$ by (4.2). Thus (4.3) becomes

$$\alpha'_\perp\left[\frac{\ddot{\Pi}(1)}{2} + \dot{\Pi}(1)\bar{\beta}\bar{\alpha}'\dot{\Pi}(1)\right]H(1) + \xi\eta'\bar{\beta}'_\perp\dot{H}(1) = g(1)\alpha'_\perp. \tag{4.4}$$

Pre and post multiplying (4.4) respectively by $\xi'_\perp$ and $\bar{\alpha}_2$, we see that

$$\alpha'_2\theta\beta_2\psi = g(1)I_{p-r-s}. \tag{4.5}$$

Then $|\psi| \neq 0$, $|\alpha'_2\theta\beta_2| \neq 0$ and $\psi = g(1)(\alpha'_2\theta\beta_2)^{-1}$ follow from $g(1) \neq 0$ and the $I(2)$ condition is satisfied.

Assume now the $I(2)$ condition holds and suppose $d = m - a > 2$; (4.5) becomes

$$\alpha'_2\theta\beta_2\psi = 0_{p-r-s}$$

and $\psi \neq 0_{p-r-s}$ contradicts $|\alpha'_2\theta\beta_2| \neq 0$ and implies $m - a = 2$. ■

These two equivalences allow us to understand the standard $I(1)$ and $I(2)$ conditions as imposing a particular shape to the adjoint matrix polynomial, which in turn ensures that the pole at the unit root has order one or two; in these cases the principal part of the Laurent expansion of (3.2) around $z = 1$ consists of one or two terms and translates into a moving average representation which involves the cumulation (or the double cumulation) of the stationary process that generates the stochastic trends.

## 5. Polynomial cointegration

The results of the previous section can thus be interpreted as alternative proofs of the Granger Representation Theorem in the $I(1)$ and $I(2)$ cases: the order of integration is established by Theorem 3.3, the explicit expression for $H(1)$ indicates the directions in which cointegration is to be found, and the restrictions implied by (4.1) and its derivatives define the (polynomial) cointegration properties of the process. In the $I(2)$ case, for example, we write the inverse of $\Pi(z)$ as

$$\Pi(z)^{-1} = \frac{H(z)}{(z-1)^2 g(z)}, \; z \neq \{1, \cdots, z_{n_r}\},$$

where $H(1) \neq 0_p$ and $g(1) \neq 0$, and $H(z)$ as

$$H(z) = H(1) + \dot{H}(1)(z-1) + (z-1)^2 H_2(z).$$

From the calculations in the proof of Proposition 4.2, we have that

$$H(1) = \beta_\perp \eta_\perp \psi \xi_\perp' \alpha_\perp', \; |\psi| \neq 0, \tag{5.1}$$

$$\beta' \dot{H}(1) = \bar{\alpha}' \dot{\Pi}(1) H(1). \tag{5.2}$$

Thus the polynomial cointegration properties of the process are the following:

**Proposition 5.1.** *Let $X_t$ be $I(2)$ and $\beta_1 = \bar{\beta}_\perp \eta$; then $\beta' X_t$ and $\beta_1' X_t$ are $I(1)$, and $\beta' X_t + \bar{\alpha}' \dot{\Pi}(1) \Delta X_t$ is $I(0)$.*

Proof. From (5.1) we have that $\beta' H(1) = 0_{r\times p}$ and $\beta_1' H(1) = 0_{s\times p}$; thus the functions $\beta' \Pi(z)^{-1}$ and $\beta_1' \Pi(z)^{-1}$ have a pole of order one at $z = 1$ and correspond to $\beta' X_t$ and $\beta_1' X_t$ being $I(1)$. Using (5.2) it is easy to see that the function $\left\{ \beta' + (1-z)\bar{\alpha}' \dot{\Pi}(1) \right\} \Pi(z)^{-1}$ has no pole at $z = 1$ and corresponds to $\beta' X_t + \bar{\alpha}' \dot{\Pi}(1) \Delta X_t$ being $I(0)$. ■

The only polynomial cointegrating relation that involves the levels and reduces the order of integration from two to zero is:

$$\beta' X_t + \bar{\alpha}' \dot{\Pi}(1) \Delta X_t.$$

Note that (5.1) and $I_p = \bar{\beta}\beta' + \bar{\beta}_1\beta_1' + \bar{\beta}_2\beta_2'$ imply $H(1) = \bar{\beta}_2\beta_2' H(1)$; thus the coefficient of the pole in $(1-z)\Pi(z)$ is actually $\bar{\beta}_2\beta_2' H(1)$. This means that there are terms in $\bar{\alpha}'\dot{\Pi}(1)\Delta X_t$ which do not help eliminate the non stationarity of $\beta' X_t$ and thus are not needed. Thus the minimal choice is to take

$$\beta' X_t + \bar{\alpha}' \dot{\Pi}(1) \bar{\beta}_2 \beta_2' \Delta X_t$$

as in Johansen (1992).

Now consider a process integrated of order $d$. The Taylor expansion of $\Pi(z)$ is

$$\Pi(z) = \sum_{n=0}^{d-1} \frac{\Pi^{(n)}(1)}{n!}(z-1)^n + (1-z)^d \Pi_d(z)$$

and translates into

$$\sum_{n=0}^{d-1} \Pi^{(n)}(1) \frac{(-1)^n}{n!} \Delta^n X_t + \Pi_d(L) \Delta^d X_t = \epsilon_t$$

in terms of the stochastic process. Since $\Delta^d X_t$ is $I(0)$ and $\Pi_d(L)$ is a finite order polynomial, $\Pi_d(L)\Delta^d X_t$ is stationary.

Then $\sum_{n=0}^{d-1} \Pi^{(n)}(1) \frac{(-1)^n}{n!} \Delta^n X_t$ is also stationary. The polynomial cointegrating relation that involves the levels and reduces the order of integration from $d$ to zero is simply

$$\beta' X_t + \bar{\alpha}' \dot{\Pi}(1) \Delta X_t - \bar{\alpha}' \frac{\ddot{\Pi}(1)}{2} \Delta^2 X_t + \cdots - (-1)^{d-1} \bar{\alpha}' \frac{\Pi^{(d-1)}(1)}{(d-1)!} \Delta^{d-1} X_t.$$

The difficulties arise when we want to find the minimal representation (see la Cour, 1998, for the $I(3)$ case). Further investigation is still needed to find a tractable solution in the general case.

## 6. Conclusion

This paper has extended the way we understand the order of integration in the univariate case to vector autoregressive processes. It has shown that there exists a very natural representation of the inverse of the characteristic polynomial, in which the order of the pole at the unit root is explicit. This result significantly simplifies the proof of the Granger Representation Theorem in the $I(1)$ and $I(2)$ cases.

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