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Key Points:

- The essential conditions and physics for various inclusion-based EMT to be consistent with the Gassmann theory are analyzed
- The inconsistency of DEM and SCA with the Gassmann theory is mainly attributed to the random distribution of pores
- The consistency of T-matrix with the Gassmann theory is mainly attributed to the periodic distribution of pores

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Gassmann Consistency for Different Inclusion-Based Effective Medium Theories: Implications for Elastic Interactions and Poroelasticity

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Abstract By evaluating the consistency of the Gassmann theory with various inclusion-based effective medium theories, we investigate the impact of elastic interactions between ellipsoidal pores on the poroelasticity. To rule out any factors that can violate the Gassmann condition, other than elastic interactions, we first construct idealized models that contain only a single set of isolated, identical, and vertically aligned ellipsoidal pores. The numerical simulation suggests that the periodic distribution of ellipsoidal pores generate uniform pore pressure distribution, whereas random distribution of ellipsoidal pores generates heterogeneous pore pressure distributions. Then we analyze the precise conditions under which the underlying Gassmann relationship is valid for various inclusion-based models. The results reveal the following: (1) Noninteracting effective medium theories are always consistent with the Gassmann prediction, simply because the elastic interactions are ignored. (2) The elastic interactions between randomly distributed pores cause heterogeneous pore pressure that violates the essential requirement of the Gassmann theory. The differential effective medium and self-consistent approximation theories corresponding to this model thus are inconsistent with the Gassmann prediction. (3) The elastic interactions between periodically distributed pores cause uniform pore pressure; therefore, the Gassmann condition is fully satisfied. The T-matrix approach explicitly takes into account such elastic interactions and thus is consistent with the Gassmann theory. It is interesting to notice that on top of other well-known common types of heterogeneities, like pore structure or fluid heterogeneities, the distribution of pores and its associated elastic interactions can be a separate source of heterogeneity, and this makes Gassmann equations not valid anymore.

1. Introduction

The theoretical determination of effective elastic behavior of porous, cracked rocks is of considerable practical interest in many fields of Earth, environmental, and engineering science. It provides a physical basis for understanding and interpreting core measurements, sonic logs, and seismic data in terms of rock properties. In the past decades, a number of studies have been made on effective medium theories (EMT), seeking to relate the mineral composition and microstructure to the effective elastic properties of a porous composite on a macroscopic scale (Gu'eguen & Kachanov, 2011; Walsh, 1965; Watt et al., 1976; Wu, 1966; Yamamoto et al., 1981). Among them, the most popular approaches are to use the inclusion-based models to predict the overall elasticity of rocks by viewing the porous composite as a solid matrix with embedded ellipsoidal inclusions representing the pores and cracks. However, in real rocks, the notoriously irregular pore and crack geometries are almost never ellipsoidal. Although suffering from the extreme idealizations of the complexity of the pore geometry, these inclusion-based models are still widely used and certainly successful in both academia and industry. This is partly attributed to the fact that solution of elastic field for ellipsoidal inclusions in porous media can be analyzed analytically. In addition, the inclusion-based model can readily provide predictive power as well as useful analogs for elastic behavior of porous rocks. For example, they are commonly employed as rock physics templates to capture the velocity trend as a function of porosity or crack density given the pore shape or crack geometry.

Various existing inclusion-based approaches differ in their strategies for extrapolating the elastic field of a single ellipsoidal inclusion (Eshelby, 1957) to treating the elastic interactions for high concentrations of inclusions. Those inclusion-based EMT implicitly assume that the ellipsoidal pores are isolated with respect to each other. Generally, in a theoretical sense, the noninteracting approximations (NIA) that typically ignore the elastic interactions between pores and cracks are considered to yield physically meaningful estimates only at dilute concentrations (Cheng, 1993; Grechka & Kachanov, 2006a; Hudson, 1980, 1981, 1994; Kachanov, 1992; Kachanov et al., 1994; Kachanov et al., 2003; Kuster & Toksöz, 1974; Liu et al., 2000; O'Connell & Budiansky, 1974; Sayers & Kachanov, 1995; Schoenberg & Douma, 1988; Vernik & Kachanov, 2010; Walsh, 1965). To overcome the dilute limit, two popular rock physics schemes, self-consistent approximation (SCA) (Budiansky, 1965; Berryman, 1980a, 1980b, 1992; Hill, 1965; Hornby et al., 1994; Wu, 1966) and differential effective medium (DEM) theories (Berryman, 1992; Cleary et al., 1980; Hornby et al., 1994; Nishizawa, 1982; Norris, 1985; Xu, 1998; Zimmerman, 1991), have been proposed to compute the macroscopic effective elastic moduli at large concentrations of ellipsoidal inclusions, implicitly assuming that the pores distribute randomly. In contrast to DEM and SCA, which handle the elastic interactions implicitly, the T-matrix approach is used to estimate effective elastic properties of a composite by explicitly taking into account the pore and crack interactions (Jakobsen, Hudson, & Hansen, 2003; Jakobsen, Johansen, & McCann, 2003). Zhao et al. (2016) use the finite element modeling method to quantitatively investigate the influences of crack interactions due to different arrangements of cracks on the stress field distribution and overall elasticity. Based on the results of numerical simulation, they show that the stress shielding significantly increases the effective stiffness, while stress amplification considerably reduces the effective stiffness.

When the inclusions are filled with fluids, an increment of pore pressure change is induced and consequently resists the compression and stiffens the rock. In the low-frequency limit where the pore pressure is equilibrated within a representative elementary volume (REV), this effect is quantified by the Gassmann equation (Biot, 1956; Gassmann, 1951), which puts a clear constraint between the elastic moduli of dry rock and saturated rock:

$$K_{sat} = K_{dry} + \frac{\alpha^2}{(\alpha - \phi)/K_0 + \phi/K_f}, \quad (1)$$

$$\mu_{sat} = \mu_{dry}, \quad (2)$$

where K_{dry} is the effective bulk modulus of dry rock, K_{sat} is the effective bulk modulus of the rock with pore fluid, K_0 is the bulk modulus of mineral material making up the rock, K_f is the effective bulk modulus of pore fluid, ϕ is the porosity, $\alpha = 1 - K_{dry}/K_0$ is Biot's coefficient, μ_{dry} is the effective shear modulus of dry rock, and μ_{sat} is the effective shear modulus of the rock with pore fluid. The Gassmann equation can be extended to anisotropic media (Brown & Korringa, 1975; Gassmann, 1951; Mavko et al., 2009), which is expressed in compliance domain and given below as

$$s_{ijkl}^{sat} = s_{ijkl}^{dry} - \frac{(s_{ijaa}^{dry} - s_{ijaa}^0)(s_{bbkl}^{dry} - s_{bbkl}^0)}{(s_{ccdd}^{dry} - s_{ccdd}^0) + \phi(\beta_{fl} - \beta_0)}, \quad (3)$$

where s_{ijkl}^{dry} is the effective elastic compliance tensor element of dry rock, s_{ijkl}^{sat} is the effective elastic compliance element of the rock saturated with pore fluid, s_{ijkl}^0 is the effective elastic compliance element of the solid mineral, β_{fl} is the fluid compressibility equal to $1/K_{fl}$, and β_0 is the mineral compressibility.

It is important to point out that the Gassmann equation is subject to the following assumptions (Brown & Korringa, 1975; Thomsen, 1985; Han & Batzle, 2004; Mavko et al., 2009; Zimmerman, 2011): (1) there is homogeneous pore pressure distribution; (2) the fluids cannot escape and should be trapped in a closed porous system; and (3) there is no chemical interaction between the rock frame and fluids that affect elasticity. In general, for a wave passing through porous sedimentary rocks, conditions (2) and (3) can be satisfied. But the validity of condition (1) is dependent on the combined effect of factors, such as pore connectivity, permeability, viscosity, and measuring frequency (Batzle et al., 2006).

Figure 1. The schematic illustration of Gassmann consistency for inclusion-based effective medium theories under the isobaric situation.

It is worth mentioning that hydraulic connectivity is not required for pore pressure equilibration. This issue can be traced back to Eshelby (1957), in which the internal pressure inside a single isolated ellipsoidal inclusion is constant and uniform, subjected to a remotely applied stress field. In this context, if the elastic interactions between inclusions are completely ignored, the ellipsoidal pores with identical shapes and orientations will have exactly the same internal pressure (Grechka, 2009).

The Gassmann theory does not impose any restrictions on the specific pore geometry. Provided that the porous rocks satisfy the condition of isobaric fluid pressure within the REV, the Gassmann theory should always work. In the inclusion-based EMT, when the isolated ellipsoidal pores are filled with fluids, the additional pore pressure increment will change the overall stress field, thus influencing the effective elastic properties.

Hence, if the microstructure of the medium imposes local isobaric pore fluid pressure conditions, theoretically, any EMT should predict the relationship between the elastic responses of dry rock and saturated rock as that predicted by the Gassmann theory (Guéguen & Sarout, 2011; Le Ravalec & Guéguen, 1996; Suvorov & Selvadurai, 2011; Thomsen, 1985). From this perspective, although the Gassmann theory and EMT are each valid within their own set of assumptions, the predictions of the two theories coincide for specific microstructural configurations that fulfill equilibrated pore fluid pressure conditions.

To put it more specifically, as shown in Figure 1, the saturated moduli can be obtained in two different manners: (a) by using the EMT to find the effective elastic properties of the dry rock and then applying the Gassmann relationship or (b) by independently applying the EMT to the saturated case. If the results obtained by using these two different approaches are equal to each other, then the indication is that these EMT are consistent with the Gassmann theory. Given that approach (b) consists of adding saturated inclusions to a nonporous background in a way such that connectivity between inclusions is completely ignored, only when the porous medium satisfies the isobaric pore fluid conditions does the concept of Gassmann consistency become meaningful. For example, Gassmann consistency is not applicable for the case of the coexistence of unconnected compliant cracks and background porosity, because it inherently violates the pore pressure equilibration conditions.

For porous sedimentary rocks containing pore heterogeneities (i.e., coexistence of stiff background porosity and cracks), approach (a), based on the Gassmann equation, is commonly considered as the relaxed status of low-frequency limit (Guo et al., 2017), whereas approach (b), ignoring fluid communication, is commonly considered as the unrelaxed status of high-frequency limit. In this case, the elastic properties computed using approach (a) are never consistent with those computed using approach (b). However, in this study, by assuming certain kinds of idea models, the results of approach (a) and approach (b) can be comparable to each other. This is more like thought experiments, which are mainly for the purpose of treating more fundamental physics from a purely theoretical perspective.

Using theoretical derivation or a numerical test, it is easy to prove that the classical bounding methods, such as Voigt-Reuss bounds and Hashin-Shtrikman bounds (Hashin & Shtrikman, 1963), are all consistent with the Gassmann theory (Han & Batzle, 2004; Yan & Han, 2011). This can be understood, because the rigorous constraints provided by those bounding methods, without any assumptions for the geometry of their constituents, ensure that the pore pressure is equilibrated; thus, the essential conditions of Gassmann consistency are fully satisfied.

However, the physical conditions for an inclusion-based EMT to satisfy the requirement of Gassmann consistency are still ambiguous. This is especially the case for nondilute inclusion theories dealing with elastic interactions. Although the relationship between Gassmann consistency and EMT have also been previously reported by many researchers (Thomsen, 1985; Le Ravalec & Guéguen, 1996; Xu, 1998; Jakobsen, 2004; Grechka, 2009; Suvorov & Selvadurai, 2011), the link between different inclusion-based models and the physics standing behind have not been systematically investigated. The present paper will treat this question in a more detailed, comprehensive, and rigorous way.

It is worth mentioning that the Gassmann equation should not be considered as a constraint for the different EMT scenarios, since all the inclusion-based EMT scenarios have their own modeling purposes and assumptions. Alternatively speaking, if one EMT inherently does not satisfy the basic assumption of the underlying Gassmann relationship, the Gassmann equation cannot provide the low-frequency limit of the fluid saturated elastic responses.

The motivation of this paper is to identify the precise conditions under which the underlying Gassmann relationship is valid for the inclusion-based theories. Based on the critical discussions and comparisons of existing inclusion-based EMT, emphasis will be placed on revealing the essential relationship between elastic interactions and poroelasticity.

2. Comparisons of Gassmann Consistency for Different EMT

2.1. Physical Model

For a porous rock containing fluid-saturated ellipsoidal pores within the REV subjected to the applied stress, its pore pressure distribution can be affected by the geometry and orientation of the ellipsoidal pores. For unconnected pores, the orientation and compliance can exercise an impact on the pore pressure distribution; for example, the randomly oriented ellipsoidal pores subjected to the pure shear stress create unequilibrated pore pressure (Mavko & Jizba, 1991; Adelinet, Fortin, & Guéguen, 2011).

Note that the requirement of Gassmann consistency is only applicable under the isobaric pore fluid conditions. In order to focus the study on the impact of elastic interactions, we purposely rule out the influences from pore structure heterogeneity and orientation heterogeneity, by constructing the physical model including only one set of vertically aligned ellipsoidal pores, all of which are identical and identically oriented with respect to the stress field. In other words, the pore space is only composed of unconnected ellipsoidal pores with identical shape and orientation. The aspect ratio of the ellipsoidal pores is set to 0.05 to simulate the elastic behavior of a cracked rock. The host matrix is isotropic and assumed to be calcite ($K = 76.8$ GPa, $\mu = 32$ GPa). The resultant cracked rock is transversely isotropic with a horizontal symmetry axis. The volume crack density ε (Hudson, 1980; O'Connell & Budiansky, 1974) is determined by the aspect ratio and the porosity of ellipsoidal pores. The saturating fluid is assumed to be brine with a bulk modulus of 2.5 GPa and a density of 1.0 g/cm^3 . If there are no other specific instructions, all the numerical simulations in this paper will be based on this crack model.

Given that the ellipsoidal pores have identical shapes and orientation, the elastic interaction between the ellipsoidal pores associated with their spatial distribution is the only influence on the pore pressure distribution, which will be investigated in more detail later.

2.2. NIA for Dilute Concentrations

The pioneering work concerning noninteracting theory is initiated by the paper of Eshelby (1957), in which he proposed to use the interaction energy approach to calculate the effective elastic constants of a homogeneous host rock permeated with dilute concentrations of ellipsoidal pores. He suggested to formulate the solution under two boundary conditions: the constant load and the constant displacement at infinity. The effective compliance under the constant load condition and the effective stiffness under the constant displacement condition can be derived as (Nishizawa, 1982; Xu, 1998)

$$\mathbf{S}^* = \mathbf{S}^0 + \sum_{r=1}^N v^r [(\mathbf{C}^0 - \mathbf{C}^r) \mathbf{E}^r - \mathbf{C}^0]^{-1} (\mathbf{C}^r - \mathbf{C}^0) \mathbf{S}^0, \quad (4)$$

$$\mathbf{C}^* = \mathbf{C}^0 - \sum_{r=1}^N v^r \mathbf{C}^0 [(\mathbf{C}^0 - \mathbf{C}^r) \mathbf{E}^r - \mathbf{C}^0]^{-1} (\mathbf{C}^r - \mathbf{C}^0), \quad (5)$$

where $\mathbf{S}^*$ and $\mathbf{C}^*$ are the effective compliance and effective stiffness of the medium including ellipsoidal pores, respectively; $\mathbf{C}^0$ and $\mathbf{S}^0$ are the fourth-rank stiffness and compliance tensors of the host rock, respectively; v^r is the volume fraction of the inclusions ($r = 1, 2, \dots, N$); $\mathbf{C}^r$ represents the stiffness tensor of the r th inclusion; and $\mathbf{E}^r$ is the well-known Eshelby tensor of the r th inclusion, which relates the eigenstrain to the total strain (Mura, 1987). Here, the inclusions refer to ellipsoidal pores. All those denotations will be also effective for the other equations in the next section. Hereafter, we term the Eshelby's formulation under

Figure 2. The effective elastic stiffness components (a) C_{11} and (b) C_{33} as functions of crack density (crack porosity) predicted by Eshelby's formulation for stiffness-based NIA. The blue line, black asterisk, and red line represent elastic stiffness for dry rock, elastic stiffness for saturated rock, and the Brown-Korringa predictions, respectively. The aspect ratio of the crack is set to 0.05.

the constant displacement condition and the constant load condition as stiffness-based NIA and compliance-based NIA, respectively.

The effective stiffness components C_{11} and C_{33} , corresponding to the P wave propagating perpendicular and parallel to the crack plane, as functions of crack density predicted by Eshelby's formulation for stiffness-based NIA and compliance-based NIA are displayed in Figures 2 and 3, respectively.

It is clear that the saturated stiffness components C_{11} and C_{33} by stiffness-based NIA and compliance-based NIA (black asterisks) are in agreement with those predicted by applying the Brown-Korringa relations to the dry C_{11} and C_{33} (red lines), which is indicative of Gassmann consistency for stiffness-based NIA and compliance-based NIA. It is also easy to demonstrate that the saturated stiffness components C_{13} , C_{44} , and C_{66} are all consistent with the Brown-Korringa predictions. In addition, it can be noticed that stiffness-based NIA typically breaks down at high crack density, whereas compliance-based NIA seems to still produce reasonable estimates at high crack density. Comparisons of those two NIA theories have been

Figure 3. The effective elastic stiffness components (a) C_{11} and (b) C_{33} as functions of crack density (crack porosity) predicted by Eshelby's formulation for compliance-based NIA. The blue line, black asterisk, and red line represent elastic stiffness for dry rock, elastic stiffness for saturated rock, and the Brown-Korringa predictions, respectively. The aspect ratio of the crack is set to 0.05.

discussed in Gu'eguen and Kachanov (2011), who have argued that the compliance-based approach is more appropriate for capturing elastic characteristics of cracked rocks.

2.3. DEM and SCA: Model Elastic Interactions Implicitly

To overcome the dilute limit of NIA, by necessity, the elastic interactions between ellipsoidal pores are accounted for to calculate the effective elastic properties. It is worth noting that all methods accounting for elastic interactions are approximate. In SCA, the contribution of excess strain due to deformation of the ellipsoidal inclusion is still employed, but the elastic interactions of ellipsoidal pores are approximated by replacing the background medium with the as yet unknown effective medium. SCA yields the effective estimates through mixing all phases, including minerals and pores, presented in the rock, in an iterative manner. The effective stiffness tensor in the SCA model can be expressed as (Bandyopadhyay, 2009; Hornby et al., 1994)

$$\sum_{r=1}^N v_r (\mathbf{C}^r - \mathbf{C}^{SCA}) [\mathbf{I} + \mathbf{G}^r (\mathbf{C}^r - \mathbf{C}^{SCA})]^{-1} = 0. \quad (6)$$

Here $\mathbf{C}^{SCA}$ indicates the self-consistent effective elastic stiffness and $\mathbf{G}^r$ is a fourth-rank tensor given by the strain Green function integrated over the inclusion shape (Eshelby, 1957; Mura, 1987):

$$\mathbf{G}_{ijkl}^r = \frac{1}{8\pi} [\bar{\mathbf{G}}_{ikjl} + \bar{\mathbf{G}}_{jkil}], \quad (7)$$

The nonzero components of $\bar{\mathbf{G}}_{ikjl}$ (see Appendix A) for a transversely isotropic system are given by Mura (1987).

However, in SCA, embedding a pore to an effective medium already containing the pore itself implies that the interactions of pores are taken into account twice. Bruner (1976) criticized that the self-consistent method might cause an overestimate of the pore interactions. To avoid the double-counting pore interactions in SCA, one can introduce the pores into the rock sequentially, with pore $n + 1$ considered to be added into a reference medium that has the effective elastic properties with n pores. In this way, pore $n + 1$ feels the effect of n pores, but not vice versa (Jaeger et al., 2007). In DEM, a small amount of inclusions of one phase is incrementally added to a background host medium in an iterative fashion. The process is continued until the desired proportion of the constituents is reached. The change in effective elastic stiffness $d\mathbf{C}$ due to an increase of the i th component dv is (Hornby et al., 1994; Xu, 1998)

$$\frac{d}{dv} (\mathbf{C}^{DEM}(v)) = \frac{1}{1-v} (\mathbf{C}^r - \mathbf{C}^{DEM}(v)) \times [\mathbf{I} + \mathbf{G}^r (\mathbf{C}^r - \mathbf{C}^{DEM}(v))]^{-1}, \quad (8)$$

where $\mathbf{C}^{DEM}(0) = \mathbf{C}^0$. The primary conceptual difference between the DEM and SCA schemes lies in the fact that SCA treats all the phases in the composite on a symmetrical basis, whereas the DEM scheme does not treat phases on an equal footing and prefers one specific phase as the host matrix. Note that both the DEM and SCA schemes implicitly assumes that the ellipsoidal pores are randomly distributed (Xu, 1998; Yamamoto et al., 1981). Here, in the anisotropic domain, the ellipsoidal pores modeled by DEM and SCA stemming from Eshelby's theory are aligned (Hornby et al., 1994).

Figures 4 and 5 show that the saturated stiffness components C11 and C33 by SCA and DEM (red lines) are not in agreement with those predicted by applying Brown-Korringa's relations to the dry C11 and C33 computed with SCA and DEM (black lines), and the discrepancy increases with an increase of crack density. This clearly demonstrates that DEM and SCA are not consistent with the more basic Gassmann theory for the scenario of randomly distributed cracks.

2.4. T-Matrix: Characterize Elastic Interactions Explicitly

Based on the integral equation methods of quantum scattering theory, the T-matrix language explicitly accounting for the spatial arrangement of inclusions is used to compute the effective elastic properties of a porous composite. Jakobsen, Hudson, and Hansen (2003) and Jakobsen (2004) formulate the effective stiffness $\mathbf{C}_T^*$ using the T-matrix approach based on two-point statistics:

Figure 4. The effective elastic stiffness components (a) C11 and (b) C33 as functions of crack density (crack porosity) predicted by SCA. The blue line represents the elastic response of dry rock simulated by SCA. The red line and black line indicate elastic stiffness for saturated rock predicted by SCA and the Brown-Korringa relations, respectively. The aspect ratio of the crack is set to 0.05.

$$\mathbf{C}_T^* = \mathbf{C}^0 + \mathbf{T}_1 (\mathbf{I} - \mathbf{T}_1^{-1} \mathbf{X})^{-1}, \quad (9)$$

where

$$\mathbf{T}_1 = \sum_{r=1}^N v^r \mathbf{t}^r, \quad (10)$$

$$\mathbf{t}^r = \delta \mathbf{C}^r (\mathbf{I} - \mathbf{G}^r \delta \mathbf{C}^r)^{-1}, \quad (11)$$

and

$$\delta \mathbf{C}^r = \mathbf{C}^r - \mathbf{C}^0. \quad (12)$$

Here $\mathbf{X}$ gives the second-order correction for the effects of inclusion tensors:

$$\mathbf{X} = - \sum_{r=1}^N \sum_{s=1}^N v^r \mathbf{t}^r \mathbf{G}_d^{rs} v^s \mathbf{t}^s, \quad (13)$$

where $\mathbf{G}_d^{rs}$ represents the two-point interaction between the r th and s th sets of inclusions. The fourth-rank tensor $\mathbf{G}_d^{rs}$ can be obtained in the same way as $\mathbf{G}^r$ (see equation (7)) except that the aspect ratio of the

Figure 5. The effective elastic stiffness components (a) C11 and (b) C33 as functions of crack density (crack porosity) predicted by DEM. The blue line represents the elastic response of dry rock simulated by DEM. The red line and black line indicate elastic stiffness for saturated rock predicted by DEM and the Brown-Korringa relations, respectively. The aspect ratio of the crack is set to 0.05.

Figure 6. The effective elastic stiffness components (a) C_{11} , (b) C_{33} , (c) C_{13} , (d) C_{44} , and (e) C_{66} as functions of crack density (crack porosity) predicted by T-matrix. The blue line represents the elastic response of dry rock simulated by T-matrix. The black asterisk and red line indicate elastic stiffness for saturated rock predicted by T-matrix and the Brown-Korringa relations, respectively. The aspect ratio of the crack is set to 0.05, and the aspect ratio of spatial distribution is set to 1.0.

inclusion is set to the aspect ratio of spatial distribution. The definition and physical meaning of the aspect ratio of spatial distribution are described in detail in Ponte and Willis (1995), Jakobsen, Hudson, and Hansen (2003), and Zhao et al. (2016). Overall, the strength of stress amplification can be significantly enhanced when crack tips closely approach each other, while the strength of stress shielding can dominate the stress interactions when the distances of crack faces are close to zero (Hu & McMechan, 2009; Zhao et al., 2016).

Note that the increment of pore pressure is dependent on the induced elastic field by ellipsoidal pores. Therefore, the elastic interactions between ellipsoidal pores certainly affect the deformation caused by pore pressure change, which determines the resulting increase in effective stiffness. We now test the requirement of Gassmann consistency on the T-matrix modeling results as shown in Figure 6. It turns out that the saturated stiffness directly computed by T-matrix (black asterisk) matches those predicted by applying Brown-Korringa's relations to the dry stiffness computed with T-matrix (red lines).

3. Numerical Simulation of Elastic Interactions Effects on Pore Pressure Distribution

To better understand the essential conditions of Gassmann consistency for different inclusion-based EMT, in this section, we use the finite element modeling method based on the COMSOL Multiphysics software to investigate the effect of the elastic interactions (the spatial distribution of the cracks) on the pore pressure distribution of the cracked porous media. The finite element approach is employed to solve the displacement-pressure formation in the framework of Biot's poroelasticity. More implementation details concerning how this method is used to simulate the pore pressure distribution in poroelastic media can be found in Quintal et al. (2011) and Cao et al. (2019). In order to simulate a far-field strain condition (strain level is less than 10^{-6}), we apply homogeneous horizontal solid displacements (8e-7m) on lateral boundaries of the 2-D linear elastic solid ($E = 84$ GPa, $\nu = 0.32$, similar to mineral host of calcite) with ellipsoidal cracks filled with brine ($K = 2.25$ GPa). The top and bottom boundaries of the sample are confined, and no shear stresses are applied. The aspect ratio of the crack is set to 0.05.

As displayed in Figure 7, we purposely construct two models containing different spatial distributions of cracks: (a) the physics scenario described by T-matrix, where the cracks are distributed in a periodic manner, and (b) the physical scenario described by the DEM or SCA modeling scheme. It is worth mentioning that

Figure 7. Two models containing different spatial distribution of cracks: (a) vertically aligned cracks are distributed in a periodic manner, corresponding to the physical scenario described by T-matrix, and (b) vertically aligned cracks are randomly distributed in the solids, corresponding to the physical scenario described by DEM and SCA. Note that in the left of (b), the two crack tips approach each other very closely, typically suggesting that stress amplification dominate the stress interactions.

T-matrix can allow for more general distributions for the spatial distribution and orientation of ellipsoidal pores (Jakobsen, Hudson, & Hansen, 2003). For the purpose of comparison with the other inclusion-based EMT that originated from Eshelby's theory (Eshelby, 1957) in a consistent way, we restrict the ellipsoidal pores to be vertically aligned and periodically distributed in the T-matrix formalism.

Figure 8 displays the detailed stress magnitude along the dashed blue lines as sketched in Figure 7, crossing the center of the cracks in the middle of the square. The value of the stress magnitude out of the elliptical cracks represent the stress field distribution in the solid, and the stress field inside the elliptical cracks indicate the pore pressure distribution. The black circles in Figure 8 also indicate the location of the center of the cracks. As illustrated in Figure 8a, it is apparent that the pore pressure in the center of the four cracks have approximately the same value. It is worth mentioning that even the ellipsoidal cracks themselves inherently induce stress concentration or dilution to a certain degree; when they are influenced by the periodic stress interaction circumstance, they still generate the identical stress field and pore pressure distribution. However, note that the pore pressure in the two cracks near the boundary of the square slightly deviates from the pore pressure of the four cracks in the middle of the square. This is mainly because the four cracks in the middle of the square have different stress field environment compared with the other two cracks, which represent a special case close to the boundary.

By contrast, as shown in Figure 8b, the randomly distributed cracks in Figure 7b, corresponding to the physical scenario described by DEM and SCA, exhibit a heterogeneous pore pressure distribution. The pore pressure magnitude even differs from crack to crack. This is of no surprise, since the patterns of the crack distribution are prone to yield stress amplification and shielding effect. As a consequence, the random and irregular crack spacing, instead of periodical spacing in the T-matrix scenario, will create a heterogeneous stress field environment. This will naturally induce the nonuniform pore pressure distribution. Note that here we concentrate on the influences of spatial distribution on local stress interaction effects, rather than on the overall stress interaction effects. Actually, the random distributions tend to compensate stress shielding and amplification effects, which, in turn, decrease the overall elastic interaction effects (Grechka & Kachanov, 2006b; Guo et al., 2018). Regular or periodic distributions, on the other hand, enhance the overall interaction effects (Guo et al., 2018; Zhao et al., 2017). Especially for a specific distribution pattern, the strong stress amplification and shielding can dominate the overall elastic interactions effects.

It is interesting to note that in addition to many known heterogeneities, such as fluid patches, pore fabric, or rock frame heterogeneity, that generate nonuniform pore pressure (Ba et al., 2016; Pride et al., 2004; Sun et al., 2015; Zhang et al., 2019; Zhao et al., 2017), elastic interactions associated with the spatial

Figure 8. Comparison of the stress magnitude variations along the defined dashed blue line corresponding to the two models in (a) Figure 7a and (b) Figure 7b. The black circles indicate the location of the center of the cracks, where the stress magnitude inside the cracks indicates the pore pressure distribution.

distribution of ellipsoidal pores might represent another type of mechanism giving rise to heterogeneous pore pressure. Moreover, it is important to emphasize that to be consistent with the physical model to be investigated here, the fluid pressure distribution obtained in Figure 8 is associated with a nonporous background, where only one set of isolated and vertically aligned ellipsoidal pores is included. If the background porosity is hydraulically connected with the cracks, then the fluid pressure should be equilibrated in the low-frequency limit (Gassmann limit). In this sense, the pore pressure distribution obtained in Figure 8 is more representative of the one expected in the unrelaxed regime (high-frequency limit).

4. Discussions

4.1. Conditions of Gassmann Consistency

4.1.1. NIA

Figures 2 and 3 illustrate that the Gassmann relationship is valid for Eshelby's formulation for stiffness-based NIA and compliance-based NIA. Note that Eshelby's theory assume that the ellipsoidal inclusions are isolated from each other and hence disconnected without fluid communications. As a consequence, such inclusion-based approaches are often considered to simulate very high frequency saturated rock behavior appropriate to ultrasonic laboratory conditions. However, the analytical results suggest that they can be consistent with the low-frequency Gassmann theory at both dilute concentrations and high crack density. The Gassmann consistency of NIA further suggests that the pore connectivity is not the essential condition for an inclusion-based EMT to be consistent with the Gassmann relationship. Indeed, the consistency of NIA with the Gassmann prediction has been analytically proven in Grechka (2009), in which they also conclude that the disconnected porosity does not necessarily validate the Gassmann equation. In addition, Jakobsen, Johansen, and McCann (2003) also demonstrate that the first-order T-matrix approach that assumes isolated inclusions and ignores elastic interactions is also consistent with the Gassmann equation.

Strictly speaking, note that the physical scenario described by NIA does not necessarily ensure the isobaric condition. Especially at high crack density (>0.10), the intense stress interaction might violate the exact isobaric condition. Consequently, such consistency is mainly because of the fact that the NIA theory itself completely ignores the elastic interactions, even at very high crack density. Thus, the pore pressure unequilibration caused by the heterogeneous stress field is not accounted for in the poroelasticity of the NIA theory. This will be firmly demonstrated by the case of the noninteracting Kuster-Toksöz model, which

Figure 9. Comparison of C_{11} as a function of crack density (crack porosity) predicted by the T-matrix method with different aspect ratios of spatial distribution (asd). The dashed line and solid line represent the effective stiffness of dry rock and saturated rock, respectively, and the differences between them indicate the corresponding fluid saturation effects. The aspect ratio of the crack is set to 0.05.

still satisfy the Gassmann consistency condition for the scenario of randomly oriented and randomly distributed cracks under hydrostatic stress field, as will be shown later.

4.1.2. DEM and SCA

As illustrated in Figures 4 and 5, the DEM and SCA theories are not consistent with the Gassmann prediction. The inconsistency of those two schemes is commonly considered to be caused by the poor connectivity or less fluid communication (Mavko et al., 2009; Xu, 1998). Remember that DEM, SCA, and T-matrix all originate from Eshelby's theory (Eshelby, 1957), which indicates that the ellipsoidal pores are all isolated with respect to each other, and the main difference lies in the distinct expression of the manner of elastic interactions. For the specific physical model investigated here, T-matrix can be consistent with the Gassmann theory, while DEM and SCA are not. That is why we conclude that pore connectivity is not the primary reason leading to the inconsistency of DEM and SCA with the Gassmann theory.

As we have shown in Figure 7b, the vertically aligned ellipsoidal pores in DEM and SCA are assumed to be randomly spaced. The complicated elastic interactions associated with random spacing inherently cause the nonuniform distribution of stress field outside the ellipsoidal inclusions, and hence, the induced fluid pressure inside the inclusions is not completely equilibrated, as illustrated in Figure 8b. Therefore, for a medium composed of an elastic solid background containing isolated ellipsoidal cracks, we believe that the inconsistency of the DEM and SCA theories with the Gassmann prediction is largely attributed to the nonuniform pore pressure distribution associated with the complex elastic interactions due to the random spacing.

4.1.3. T-Matrix

In contrast, by explicitly taking into account the elastic interactions, the theoretical modeling result clearly suggests that T-matrix is in concert with the underlying Gassmann theory (Figure 6). This is also understandable, as illustrated in Figures 7a and 8a; when the pores are periodically distributed as described in the T-matrix formalism, all the pores are subjected to the same stress conditions and influenced by the same elastic interactions. Consequently, the induced pore pressure is identical for all the inclusions, satisfying the essential condition of the Gassmann prediction.

Figure 9 illustrates the influences of the aspect ratio of spatial distribution on the fluid saturation effects. The different spatial distributions of pores certainly cause different types of stress interactions. It is interesting to

Figure 10. Comparison of bulk modulus as a function of crack density (crack porosity) predicted by the (a) SCA and (b) DEM theories in isotropic domain. The blue line represents the elastic response of dry rock. The black line and red line indicate elastic stiffness for saturated rock predicted by EMT and the Gassmann relations, respectively. The aspect ratio of the ellipsoidal inclusion is set to 0.05.

notice that the increase in effective stiffness due to fluid saturation effects varies with the aspect ratio of spatial distribution (elastic interaction). As we can see, the saturation effect is more pronounced when the aspect ratio of spatial distribution increases. This is expected, since the crack tips will approach each other closely with the increase of the aspect ratio of spatial distribution, and consequently, the stress amplification will increase much stronger than the stress shielding. In this case, the overall strain and effective compressibility will increase accordingly, and hence, the induced pore pressure increment will exercise a bigger influence on resisting the compression and stiffening the rock.

It is necessary to point out that the T-matrix formalism applied here only deals with the two-body problems in scattering theory (Jakobsen, Hudson, & Hansen, 2003). In other words, only elastic interactions between two inclusions are accounted for, whereas the high-order elastic interactions beyond the two inclusions are explicitly ignored. Consequently, the influence of elastic interactions between pores are not completely taken into account. From this point of view, we must admit that even the T-matrix method based on a two-point correlation function is incapable of characterizing the stress-strain relationship for porous composites with all physics involved.

4.2. Gassmann Consistency for the Scenario of Randomly Oriented Cracks

In the present paper, the comparisons and discussions regarding the Gassmann consistency for EMT only include the cases of identically aligned cracks. These models are intimately associated with the classical Eshelby solution to the strain field of an ellipsoidal inclusion with a perfect orientation embedded in an infinite, homogeneous solid (Eshelby, 1957). Here we will briefly discuss the inclusion-based EMT for the scenario of randomly oriented cracks.

Conventionally, for randomly oriented cracks imposed by hydrostatic stress, the fluid pressure inside the cracks is considered to be equilibrated, and hence, no bulk dispersion is expected to be observed. However, as illustrated in Figure 8, the pore pressure distributions are not only affected by the pore heterogeneity but also influenced by the stress field imposed by the elastic interactions. Even for the medium composed of randomly oriented cracks, the elastic interaction associated with random spacing might still produce locally heterogeneous pore pressure distribution to a certain degree. This has been demonstrated in Figure 10; regarding the DEM (Berryman, 1992; Norris, 1985; Zimmerman, 1991) and SCA (Budiansky, 1965; Hill, 1965; Wu, 1966) schemes in isotropic media describing the scenario of randomly oriented cracks, it appears that the saturated bulk modulus differs from that predicted by the Gassmann theory. It can be noticed from Figure 11a that for the same physical scenario of randomly oriented cracks, the saturated bulk modulus directly computed by the noninteracting Kuster-Toksöz model (black asterisk) exactly matches those predicted by applying the Gassmann relations. Clearly, this is not because of the isobaric condition but mainly because Kuster-Toksöz model itself completely ignores the elastic interactions, which is similar to the noninteracting stiffness- and compliance-based Eshelby theory.

Figure 11. Comparison of (a) bulk modulus and (b) shear modulus as functions of crack density (crack porosity) predicted by the Kuster-Toksoz model. The blue line represents the elastic response of dry rock. The black asterisk (or line) and red line indicate elastic stiffness for saturated rock predicted by the Kuster-Toksoz model and Gassmann relations, respectively. The aspect ratio of the ellipsoidal inclusion is set to 0.05.

For a medium composed of randomly oriented cracks, the pure shear stress can induce different fluid pressures in the cracks depending on their orientation, therefore causing shear dispersion (Adelinet et al., 2011; David, 2012; Le Ravalec & Guéguen, 1996; Mavko & Jizba, 1991; Mukerji & Mavko, 1994). This has been demonstrated in Figure 11b; for the noninteracting Kuster-Toksoz model, the saturated shear modulus is distinct from the undrained shear modulus, or Gassmann-predicted shear modulus. As expected, the shear wave velocity discrepancy for the DEM and SCA modeling schemes are more significant, which will be illustrated in the next section. Both the random orientation of cracks and elastic interaction might contribute to the unequilibrated pore pressure.

4.3. Discrepancy of Using Gassmann Equation for Different EMT

For inclusion-based EMT that inherently do not satisfy the homogenous pore pressure conditions, it is apparent that the Gassmann equation is not valid anymore. Here the velocity discrepancy of using the Gassmann equation for different EMT is defined as $\frac{|V_{saturated} - V_{gassmann}|}{|V_{gassmann}|}$. Here $V_{saturated}$ is the saturated velocity directly calculated using EMT and $V_{gassmann}$ is the velocity computed based on the Gassmann (or Brown-Korrington) equation. Indeed, such velocity discrepancy is indicative of estimating the error committed if one uses the Gassmann equation to compute the low-frequency limit. Figures 12a and 12b illustrate the relationship of magnitude of P wave velocity discrepancy as a function of crack density for the DEM and SCA modeling schemes, respectively. The blue line, red line, and black line indicate the vertical P wave, horizontal P wave, and P wave in the isotropic domain, respectively. The former two are computed based on the anisotropic DEM and SCA schemes, mimicking the scenario of vertically aligned cracks; the latter one is calculated using isotropic DEM and SCA, representing the scenario of randomly oriented cracks. As expected, the magnitude of velocity discrepancy increases with an increase of crack density. This is mainly because the elastic interactions also enhance with an increase of crack density, leading to a higher degree of pore pressure unequilibration.

It can be noticed that the discrepancy magnitude of vertical P wave is much larger than that of the horizontal P wave for both the DEM and SCA modeling schemes. Indeed, this is because the P wave perpendicular to the crack plane easily creates much larger pore pressure responses (Rubino et al., 2013). By contrast, pore pressure change is quite small in response to the passing P wave parallel to the crack plane. It is also interesting to notice that the P wave discrepancy for the randomly oriented cracks falls between the vertical P wave and the horizontal P wave.

Additionally, in comparison with the DEM and SCA modeling schemes, it is found that the P wave discrepancy magnitude for SCA is much larger than that of DEM. This is probably attributed to the overestimate of

Figure 12. The relationship of P wave velocity discrepancy as a function of crack density for the (a) DEM and (b) SCA modeling schemes. The blue line, red line, and black line indicate the vertical P wave, horizontal P wave, and P wave in the isotropic domain, respectively.

the pore interactions by SCA, which double-counts the pore interactions (Bruner, 1976), while DEM only takes into account the elastic interactions once (Guégun et al., 1997). Such an overestimate of the pore interactions might enhance the heterogeneous distribution of pore pressure. This also implies that velocity discrepancy deviating from the Gassmann prediction is not merely attributed to the physical scenario of the orientation and distribution patterns of pores but also closely related to the manner of the elastic interactions in their modeling schemes.

Similarly, Figure 13 displays the comparison of the apparent S wave discrepancy magnitude as a function of crack density for the three types of modeling methods: DEM, SCA, and the Kuster-Toksöz model. It can be noted that the scenario of the randomly oriented cracks submitted to the shear stress induces a much larger apparent S wave discrepancy magnitude than does the scenario of the aligned cracks. Moreover, the noninteracting Kuster-Toksöz model generates a smaller shear wave discrepancy that is merely related to the random orientation of cracks, but the interacting isotropic DEM or SCA does not. Such a discrepancy between them probably stems from the pore pressure gradient associated with elastic interactions.

Figure 13. The relationship of the apparent S wave velocity discrepancy as a function of crack density for the three types of modeling methods: DEM, SCA, and Kuster-Toksöz (KT) model. The blue line, red line, and black line indicate the S wave polarized in the vertical direction, horizontal direction, and isotropic domain, respectively. The green line indicates the S wave calculated using the noninteracting Kuster-Toksöz model for the scenario of randomly oriented cracks.

Figure 14. Variation of the elastic stiffness C_{11} predicted by NIA with increasing crack density. The solid matrix is calcite, and the matrix (stiff) porosity is 0.2. The aspect ratio of the matrix porosity and cracks are set to 0.7 and 0.05, respectively. The blue line represents stiffness of dry rock, the red line represents stiffness predicted by the Brown-Korringa relations, and the black line indicates the brine-saturated stiffness directly simulated by DEM.

previously discussed, since no pore heterogeneities are present at this initial status. It is worth mentioning that we set the aspect ratio of matrix pores as 0.2 (relatively small compared with the pore heterogeneities cases for NIA and T-matrix) to ensure that the elastic interaction is strong enough to induce the heterogeneous pore pressure distribution, because the effect of elastic interactions is significantly mitigated as the

Figure 15. Variation of the elastic stiffness C_{11} predicted by DEM with increasing crack density. The solid matrix is calcite, and the matrix (stiff) porosity is 0.2. The aspect ratio of the matrix porosity and cracks are set to 0.2 and 0.05, respectively. The blue line represents stiffness of dry rock, the red line represents stiffness predicted by the Brown-Korringa relations, and the black line indicates the brine-saturated stiffness directly simulated by DEM.

4.4. Cases of Coexistence of Stiff and Compliant Porosity

Note that all the ellipsoidal pores have identical geometry in the abovementioned scenarios; therefore, the elastic heterogeneities in pore stiffness are not included. For pore space composed of stiff and compliant porosity, the inherent limitation of isolated ellipsoidal pores in inclusion-based EMT make the unequilibrated pore pressure unable to be relaxed. Since the cracks are much more compliant than are the pores, the induced fluid pressure in those two types of pores are significantly different, and this fluid pressure equalization is clearly a time-dependent process.

For NIA (we take stiffness-based NIA as an example), Figure 14 illustrates that, in the case of coexistence of stiff matrix porosity compliant and crack porosity and (the aspect ratio of the matrix porosity and cracks are set to 0.7 and 0.05, respectively), the saturated stiffness (C_{11}) directly estimated by NIA significantly deviates from the Brown-Korringa prediction. Indeed, Thomsen (1985) has analytically proven that NIA is inconsistent with the Gassmann theory for the porous rock containing pore heterogeneities. Similar for NIA, for EMT implicitly accounting for the elastic interactions (we take DEM as an example), as displayed in Figure 15, it is apparent that there exists a significant discrepancy between the Brown-Korringa prediction and the saturated stiffness simulated by DEM. As expected, the difference at the crack density of zero is caused by the unequilibrated pore pressure associated with elastic interactions as

As shown in Figure 16, the saturated stiffness directly simulated by T-matrix clearly deviates from the prediction by the Brown-Korringa's relations when both stiff pores and soft cracks are present. This suggests that when the isobaric pore fluid condition cannot be satisfied, T-matrix is not consistent with the Gassmann theory anymore. It is necessary to point out that the green line indicates the saturated stiffness computed using the hybrid modeling scheme proposed by Barbosa, Rubino, Caspari, Gurevich, et al. (2017) and Guo et al. (2017), where the background porosity is saturated using the Gassmann equation first and then the fracture porosity is saturated using the Gassmann equation again in two steps. It is interesting to note that the high-frequency saturated stiffness simulated by the hybrid modeling scheme is higher than that simulated by simultaneously incorporating the saturated background and fracture porosity using T-matrix.

Such a discrepancy may come from two aspects: (1) To incorporate the saturated background and crack porosity simultaneously, the elastic interactions between the background and crack porosity are explicitly accounted for in computing the dry and saturated stiffness using T-matrix. However, for the hybrid modeling schemes, in each

Figure 16. Variation of the elastic stiffness C_{11} predicted by T-matrix with increasing crack density. The solid matrix is calcite, and the matrix (stiff) porosity is 0.2. The aspect ratio of the matrix porosity and cracks are set to 0.7 and 0.05, respectively. The aspect ratio of spatial distribution is defined as 1.0 for T-matrix modeling. The blue line represents stiffness of dry rock, the red line represents stiffness predicted by the Brown-Korrington relations, and the black line indicates the brine-saturated stiffness directly simulated by T-matrix. The green line indicates the saturated stiffness computed using the hybrid modeling scheme proposed by Barbosa, Rubino, Caspari, Gurevich, et al. (2017) and Guo et al. (2017), where the background porosity is saturated using the Gassmann equation first and then the fracture porosity is saturated using the Gassmann equation again in two steps.

step for computing the elasticity of dry rock, only the elastic interactions between the matrix pores or crack itself at each step are accounted for. The different manners of handling elastic interactions might affect the pore pressure distribution and hence the elastic behavior of the saturated rock. (2) The fluid saturation effects based on the Gassmann equation are strongly dependent on the interaction of “dry rock” properties, “solid grain” properties, and porosity. Especially, the fluid saturation effects are significantly enhanced for a purely cracked rock due to its relatively soft matrix properties and very low porosity. This might explain why the hybrid modeling schemes yield the higher estimation for the unrelaxed status. The modeling results here also suggest that the saturating strategies for inclusion-based EMT significantly affect the prediction of elastic properties corresponding to the unrelaxed status.

For a detailed analysis of the effect of hydraulic connectivity and frequency-dependent elastic responses due to the fluid pressure diffusion between stiff matrix porosity and compliant fractures, see Thomsen (1995), Le Ravalec and Guéguen (1996), Chapman et al. (2002), Gurevich (2003), Rubino et al. (2014), Barbosa, Rubino, Caspari, and Holliger (2017), Ba et al. (2017), Barbosa et al. (2018), and Guo et al. (2018).

5. Concluding Remarks

Through the preceding comparisons and discussions on the various EMT, as well as the numerical simulation concerning the impact of elastic interactions on the pore pressure distribution, we have analyzed the physical conditions under which the inclusion-based models are consistent with the underlying Gassmann relationship. For the elastic medium composed of a single set of vertically aligned and identical ellipsoidal pores (having the same pore stiffness), the key issues for an EMT to be consistent with the Gassmann prediction might lie in two aspects: (1) The noninteracting EMT methods (e.g., NIA and the

Kuster-Toksoz model) are basically consistent with the Gassmann prediction, which is mainly due to the fact that the complex elastic interaction effects are completely ignored in their theoretical considerations; and (2) for EMT accounting for elastic interactions, if the distribution patterns of ellipsoidal pores satisfy the isobaric condition (e.g., periodically distributed cracks), then the corresponding EMT (e.g., the T-matrix method) can be consistent with the Gassmann theory. Otherwise, for randomly distributed ellipsoidal pores described by the modeling schemes of DEM and SCA, the unequilibrated pore pressure caused by the elastic interactions will lead to the inconsistency of the Gassmann prediction. Also, in addition to the many known factors (e.g., pore fabric, fluids, and rock frame heterogeneity), it is found that the elastic interactions associated with the spatial distribution of pores can also cause unequilibrated pore pressure, making the Gassmann equation not applicable anymore.

However, this does not mean that the T-matrix approach is superior to other EMT. Because it is often difficult to quantify the parameter concerning the spatial distribution of cracks or pores, even on a statistical basis, it is important to point out that each effective medium theory has its own advantage and disadvantage. Different EMT should be chosen according to the corresponding geological configurations or specific purposes.

Appendix A: Evaluation of Tensor $\bar{G}_{ijkl}$

The tensor $\bar{G}_{ijkl}$ used to calculate $\mathbf{G}_{ijkl}$ tensor for the transversely isotropic media in equation (7) is given by Nishizawa (1982) and Mura (1987). It has 12 nonzero elements, and they are shown by the following integral form:

$$\begin{aligned}\bar{G}_{1111} &= \bar{G}_{2222} = \frac{\pi}{2} \int_0^1 \Delta(1-x^2) \{ [f(1-x^2) + h\rho^2 x^2] [(3e+d)(1-x^2) + 4f\rho^2 x^2] - g^2 \rho^2 x^2 (1-x^2) \} dx \\ \bar{G}_{3333} &= 4\pi \int_0^1 \Delta \rho^2 x^2 \{ [d(1-x^2) + f\rho^2 x^2] [e(1-x^2) + f\rho^2 x^2] \\ \bar{G}_{1122} &= \bar{G}_{2211} = \frac{\pi}{2} \int_0^1 \Delta(1-x^2) \{ [f(1-x^2) + h\rho^2 x^2] [(e+3d)(1-x^2) + 4f\rho^2 x^2] - 3g^2 \rho^2 x^2 (1-x^2) \} dx \\ \bar{G}_{1133} &= \bar{G}_{2233} = 2\pi \int_0^1 \Delta \rho^2 x^2 \{ [(d+e)(1-x^2) + 2f\rho^2 x^2] [f(1-x^2) + h\rho^2 x^2] - g^2 \rho^2 x^2 (1-x^2) \} dx \\ \bar{G}_{3311} &= \bar{G}_{3322} = 2\pi \int_0^1 \Delta(1-x^2) [d(1-x^2) + f\rho^2 x^2] [e(1-x^2) + f\rho^2 x^2] dx \\ \bar{G}_{1212} &= \bar{G}_{2121} = \frac{\pi}{2} \int_0^1 \Delta(1-x^2)^2 \{ g^2 \rho^2 x^2 - (d-e)[f(1-x^2) + h\rho^2 x^2] \} dx \\ \bar{G}_{1313} &= \bar{G}_{2323} = -2\pi \int_0^1 \Delta g \rho^2 x^2 (1-x^2) [e(1-x^2) + f\rho^2 x^2] dx\end{aligned}$$

where

$$\begin{aligned}d &= c_{11}, \\ e &= (c_{11} - c_{12})/2, \\ f &= c_{44}, \\ g &= c_{13} + c_{44}, \\ h &= c_{33},\end{aligned}$$

$$\Delta^{-1} = [e(1-x^2) + f\rho^2 x^2] \{ [d(1-x^2) + f\rho^2 x^2] [f(1-x^2) + h\rho^2 x^2] - g^2 \rho^2 x^2 (1-x^2) \},$$

c_{ij} are elastic tensor elements of the host rock in Voigt notation, and p is the inverse of aspect ratio $\rho = 1/\alpha$.

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