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## GORENSTEIN DERIVED FUNCTORS

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ABSTRACT. Over any associative ring  $R$  it is standard to derive  $\text{Hom}_R(-, -)$  using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains  $\text{Ext}_R^n(-, -)$  in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product  $- \otimes_R -$  using Gorenstein flat modules.

### 1. INTRODUCTION

When  $R$  is a two-sided Noetherian ring, Auslander and Bridger [2] introduced in 1969 the G-dimension,  $\text{G-dim}_R M$ , for every *finite* (that is, finitely generated)  $R$ -module  $M$ . They proved the inequality  $\text{G-dim}_R M \leq \text{pd}_R M$ , with equality  $\text{G-dim}_R M = \text{pd}_R M$  when  $\text{pd}_R M < \infty$ , along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called *Gorenstein projectives*. Over a general ring  $R$ , Enochs and Jenda in [6] defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if  $R$  is two-sided Noetherian, and  $G$  is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following [4, Theorem (4.2.6)]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary  $R$ -modules. At this point we need to introduce:

**1.1 (Notation).** Throughout this paper, we use the following notation:

- $R$  is an associative ring. All modules are—if not specified otherwise—*left*  $R$ -modules, and the category of all  $R$ -modules is denoted  $\mathcal{M}$ . We use  $\mathcal{A}$  for the category of abelian groups (that is,  $\mathbb{Z}$ -modules).
- We use  $\mathcal{GP}$ ,  $\mathcal{GI}$  and  $\mathcal{GF}$  for the categories of *Gorenstein projective*, *Gorenstein injective* and *Gorenstein flat*  $R$ -modules; please see [6] and [8], or Definition 2.7 below.
- Furthermore, for each  $R$ -module  $M$  we write  $\text{Gpd}_R M$ ,  $\text{Gid}_R M$  and  $\text{Gfd}_R M$  for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of  $M$ , respectively.

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Now, given our base ring  $R$ , the usual right derived functors  $\text{Ext}_R^n(-, -)$  of  $\text{Hom}_R(-, -)$  are important in homological studies of  $R$ . The material presented here deals with the Gorenstein right derived functors  $\text{Ext}_{\mathcal{GP}}^n(-, -)$  and  $\text{Ext}_{\mathcal{GI}}^n(-, -)$  of  $\text{Hom}_R(-, -)$ .

More precisely, let  $N$  be a fixed  $R$ -module. For an  $R$ -module  $M$  that has a *proper left  $\mathcal{GP}$ -resolution*  $\mathbf{G} = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow 0$  (please see 2.1 below for the definition of proper resolutions), we define

$$\text{Ext}_{\mathcal{GP}}^n(M, N) := \text{H}^n(\text{Hom}_R(\mathbf{G}, N)).$$

From 2.4 it will follow that  $\text{Ext}_{\mathcal{GP}}^n(-, N)$  is a well-defined contravariant functor, defined on the full subcategory,  $\text{LeftRes}_{\mathcal{M}}(\mathcal{GP})$ , of  $\mathcal{M}$ , consisting of all  $R$ -modules that have a proper left  $\mathcal{GP}$ -resolution.

For a fixed  $R$ -module  $M'$  there is a similar definition of the functor  $\text{Ext}_{\mathcal{GI}}^n(M', -)$ , which is defined on the full subcategory,  $\text{RightRes}_{\mathcal{M}}(\mathcal{GI})$ , of  $\mathcal{M}$ , consisting of all  $R$ -modules that which have a proper right  $\mathcal{GI}$ -resolution. Now, the best one could *hope* for is the existence of isomorphisms,

$$\text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_{\mathcal{GI}}^n(M, N),$$

which are functorial in each variable  $M \in \text{LeftRes}_{\mathcal{M}}(\mathcal{GP})$  and  $N \in \text{RightRes}_{\mathcal{M}}(\mathcal{GI})$ . The aim of this paper is to show a slightly weaker result.

When  $R$  is  $n$ -Gorenstein (meaning that  $R$  is both left and right Noetherian, with self-injective dimension  $\leq n$  from both sides), Enochs and Jenda [9, Theorem 12.1.4] have proved the existence of such functorial isomorphisms  $\text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_{\mathcal{GI}}^n(M, N)$  for all  $R$ -modules  $M$  and  $N$ .

It is important to note that for an  $n$ -Gorenstein ring  $R$ , we have  $\text{Gpd}_R M < \infty$ ,  $\text{Gid}_R M < \infty$ , and also  $\text{Gfd}_R M < \infty$  for all  $R$ -modules  $M$ ; please see [9, Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring  $R$ , [12, Proposition 2.18] (which is restated in this paper as Proposition 3.1) implies that the category  $\text{LeftRes}_{\mathcal{M}}(\mathcal{GP})$  contains all  $R$ -modules  $M$  with  $\text{Gpd}_R M < \infty$ ; that is, every  $R$ -module with finite  $\text{G}$ -projective dimension has a proper left  $\mathcal{GP}$ -resolution. Also, every  $R$ -module with finite  $\text{G}$ -injective dimension has a proper right  $\mathcal{GI}$ -resolution. So  $\text{RightRes}_{\mathcal{M}}(\mathcal{GI})$  contains all  $R$ -modules  $N$  with  $\text{Gid}_R N < \infty$ .

Theorem 3.6 in this text proves that the functorial isomorphisms  $\text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_{\mathcal{GI}}^n(M, N)$  hold over *arbitrary* rings  $R$ , provided that  $\text{Gpd}_R M < \infty$  and  $\text{Gid}_R N < \infty$ . By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product  $- \otimes_R -$ , using proper left  $\mathcal{GP}$ -resolutions and proper left  $\mathcal{GF}$ -resolutions. This has also been proved by Enochs and Jenda [9, Theorem 12.2.2] in the case when  $R$  is  $n$ -Gorenstein.

## 2. PRELIMINARIES

Let  $T: \mathcal{C} \rightarrow \mathcal{E}$  be any additive functor between abelian categories. One usually derives  $T$  using resolutions consisting of projective or injective objects (if the category  $\mathcal{C}$  has enough projectives or injectives). This section is a very brief note on how to derive functors  $T$  with resolutions consisting of objects in some subcategory  $\mathcal{X} \subseteq \mathcal{C}$ . The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.

**2.1 (Proper Resolutions).** Let  $\mathcal{X} \subseteq \mathcal{C}$  be a full subcategory. A *proper left  $\mathcal{X}$ -resolution* of  $M \in \mathcal{C}$  is a complex  $\mathbf{X} = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0$  where  $X_i \in \mathcal{X}$ , together with a morphism  $X_0 \rightarrow M$ , such that  $\mathbf{X}^+ := \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$  is also a complex, and such that the sequence

$$\mathrm{Hom}_{\mathcal{C}}(X, \mathbf{X}^+) = \cdots \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, X_1) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, X_0) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, M) \rightarrow 0$$

is exact for every  $X \in \mathcal{X}$ . We sometimes refer to  $\mathbf{X}^+ = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$  as an *augmented proper left  $\mathcal{X}$ -resolution*. We do not require that  $\mathbf{X}^+$  itself is exact. Furthermore, we use  $\mathrm{LeftRes}_{\mathcal{C}}(\mathcal{X})$  to denote the full subcategory of  $\mathcal{C}$  consisting of those objects that have a proper left  $\mathcal{X}$ -resolution. Note that  $\mathcal{X}$  is a subcategory of  $\mathrm{LeftRes}_{\mathcal{C}}(\mathcal{X})$ .

*Proper right  $\mathcal{X}$ -resolutions* are defined dually, and the full subcategory of  $\mathcal{C}$  consisting of those objects that have a proper right  $\mathcal{X}$ -resolution is  $\mathrm{RightRes}_{\mathcal{C}}(\mathcal{X})$ .

The importance of working with *proper* resolutions comes from the following:

**Proposition 2.2.** *Let  $f: M \rightarrow M'$  be a morphism in  $\mathcal{C}$ , and consider the diagram*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ \cdots & \longrightarrow & X'_2 & \longrightarrow & X'_1 & \longrightarrow & X'_0 & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

where the upper row is a complex with  $X_n \in \mathcal{X}$  for all  $n \geq 0$ , and the lower row is an augmented proper left  $\mathcal{X}$ -resolution of  $M'$ . Then the following conclusions hold:

- (i) There exist morphisms  $f_n: X_n \rightarrow X'_n$  for all  $n \geq 0$ , making the diagram above commutative. The chain map  $\{f_n\}_{n \geq 0}$  is called a *lift of  $f$* .
- (ii) If  $\{f'_n\}_{n \geq 0}$  is another lift of  $f$ , then the chain maps  $\{f_n\}_{n \geq 0}$  and  $\{f'_n\}_{n \geq 0}$  are homotopic.

*Proof.* The proof is an exercise; please see [9, Exercise 8.1.2]. □

*Remark 2.3.* A few comments are in order:

- In our applications, the class  $\mathcal{X}$  contains all projectives. Consequently, all the augmented proper left  $\mathcal{X}$ -resolutions occurring in this paper will be exact. Also, all augmented proper right  $\mathcal{Y}$ -resolutions will be exact, when  $\mathcal{Y}$  is a class of  $R$ -modules containing all injectives.
- Recall (see [15, Definition 1.2.2]) that an  $\mathcal{X}$ -precover of  $M \in \mathcal{C}$  is a morphism  $\varphi: X \rightarrow M$ , where  $X \in \mathcal{X}$ , such that the sequence

$$\mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X', \varphi)} \mathrm{Hom}_{\mathcal{C}}(X', M) \longrightarrow 0$$

is exact for every  $X' \in \mathcal{X}$ . Hence, in an augmented proper left  $\mathcal{X}$ -resolution  $\mathbf{X}^+$  of  $M$ , the morphisms  $X_{i+1} \rightarrow \mathrm{Ker}(X_i \rightarrow X_{i-1})$ ,  $i > 0$ , and  $X_0 \rightarrow M$  are  $\mathcal{X}$ -precovers.

- What we have called *proper  $\mathcal{X}$ -resolutions*, Enochs and Jenda [9, Definition 8.1.2] simply call  $\mathcal{X}$ -resolutions. We have adopted the terminology *proper* from [3, Section 4].

**2.4 (Derived Functors).** Consider an additive functor  $T: \mathcal{C} \rightarrow \mathcal{E}$  between abelian categories. Let us assume that  $T$  is covariant, say. Then (as usual) we can define the  $n^{\mathrm{th}}$  left derived functor

$$L_n^{\mathcal{X}}T: \mathrm{LeftRes}_{\mathcal{C}}(\mathcal{X}) \rightarrow \mathcal{E}$$

of  $T$ , with respect to the class  $\mathcal{X}$ , by setting  $L_n^{\mathcal{X}}T(M) = H_n(T(\mathbf{X}))$ , where  $\mathbf{X}$  is any proper left  $\mathcal{X}$ -resolution of  $M \in \text{LeftRes}_{\mathcal{C}}(\mathcal{X})$ . Similarly, the  $n^{\text{th}}$  right derived functor

$$R_{\mathcal{X}}^n T: \text{RightRes}_{\mathcal{C}}(\mathcal{X}) \rightarrow \mathcal{E}$$

of  $T$  with respect to  $\mathcal{X}$  is defined by  $R_{\mathcal{X}}^n T(N) = H_n(T(\mathbf{Y}))$ , where  $\mathbf{Y}$  is any proper right  $\mathcal{X}$ -resolution of  $N \in \text{RightRes}_{\mathcal{C}}(\mathcal{X})$ . These constructions are well-defined and functorial in the arguments  $M$  and  $N$  by Proposition 2.2.

The situation where  $T$  is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

**2.5 (Balanced Functors).** Next we consider yet another abelian category  $\mathcal{D}$ , together with a full subcategory  $\mathcal{Y} \subseteq \mathcal{D}$  and an additive functor  $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  in *two* variables. We will assume that  $F$  is contravariant in the first variable, and covariant in the second variable.

*Actually, the variance of the variables of  $F$  is not important, and the definitions and results below can easily be modified to fit the situation where  $F$  is covariant in both variables, say.*

For fixed  $M \in \mathcal{C}$  and  $N \in \mathcal{D}$  we can then consider the two right derived functors as in 2.4:

$$R_{\mathcal{X}}^n F(-, N): \text{LeftRes}_{\mathcal{C}}(\mathcal{X}) \rightarrow \mathcal{E} \quad \text{and} \quad R_{\mathcal{Y}}^n F(M, -): \text{RightRes}_{\mathcal{D}}(\mathcal{Y}) \rightarrow \mathcal{E}.$$

If furthermore  $M \in \text{LeftRes}_{\mathcal{C}}(\mathcal{X})$  and  $N \in \text{RightRes}_{\mathcal{D}}(\mathcal{Y})$ , we can ask for a sufficient condition to ensure that

$$R_{\mathcal{X}}^n F(M, N) \cong R_{\mathcal{Y}}^n F(M, N),$$

functorial in  $M$  and  $N$ . Here we wrote  $R_{\mathcal{X}}^n F(M, N)$  for the functor  $R_{\mathcal{X}}^n F(-, N)$  applied to  $M$ . Another, and perhaps better, notation could be

$$R_{\mathcal{X}}^n F(-, N)[M].$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term *left/right balanced functor* (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

**Theorem 2.6.** *Consider the functor  $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  which is contravariant in the first variable and covariant in the second variable, together with the full subcategories  $\mathcal{X} \subseteq \mathcal{C}$  and  $\mathcal{Y} \subseteq \mathcal{D}$ . Assume that we have full subcategories  $\tilde{\mathcal{X}}$  and  $\tilde{\mathcal{Y}}$  of  $\text{LeftRes}_{\mathcal{C}}(\mathcal{X})$  and  $\text{RightRes}_{\mathcal{D}}(\mathcal{Y})$ , respectively, satisfying:*

- (i)  $\mathcal{X} \subseteq \tilde{\mathcal{X}}$  and  $\mathcal{Y} \subseteq \tilde{\mathcal{Y}}$ .
- (ii) *Every  $M \in \tilde{\mathcal{X}}$  has an augmented proper left  $\mathcal{X}$ -resolution  $\dots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ , such that  $0 \rightarrow F(M, Y) \rightarrow F(X_0, Y) \rightarrow F(X_1, Y) \rightarrow \dots$  is exact for all  $Y \in \mathcal{Y}$ .*
- (iii) *Every  $N \in \tilde{\mathcal{Y}}$  has an augmented proper right  $\mathcal{Y}$ -resolution  $0 \rightarrow N \rightarrow Y^0 \rightarrow Y^1 \rightarrow \dots$ , such that  $0 \rightarrow F(X, N) \rightarrow F(X, Y^0) \rightarrow F(X, Y^1) \rightarrow \dots$  is exact for all  $X \in \mathcal{X}$ .*

*Then we have functorial isomorphisms*

$$R_{\mathcal{X}}^n F(M, N) \cong R_{\mathcal{Y}}^n F(M, N),$$

*for all  $M \in \tilde{\mathcal{X}}$  and  $N \in \tilde{\mathcal{Y}}$ .*

*Proof.* Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [9, Proposition 8.2.14], or the proofs of [14, Theorems 2.7.2 and 2.7.6].  $\square$

In the next paragraphs we apply the results above to special categories  $\mathcal{X}$ ,  $\tilde{\mathcal{X}}$ ,  $\mathcal{C}$  and  $\mathcal{Y}$ ,  $\tilde{\mathcal{Y}}$ ,  $\mathcal{D}$ , including the categories mentioned in 1.1. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A *complete projective resolution* is an exact sequence of projective modules,

$$\mathbf{P} = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow \cdots,$$

such that  $\text{Hom}_R(\mathbf{P}, Q)$  is exact for every projective  $R$ -module  $Q$ . An  $R$ -module  $M$  is called *Gorenstein projective* ( $G$ -projective for short), if there exists a complete projective resolution  $\mathbf{P}$  with  $M \cong \text{Im}(P_0 \rightarrow P_{-1})$ . *Gorenstein injective* ( $G$ -injective for short) modules are defined dually.

A *complete flat resolution* is an exact sequence of flat (left)  $R$ -modules,

$$\mathbf{F} = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots,$$

such that  $I \otimes_R \mathbf{F}$  is exact for every injective right  $R$ -module  $I$ . An  $R$ -module  $M$  is called *Gorenstein flat* ( $G$ -flat for short), if there exists a complete flat resolution  $\mathbf{F}$  with  $M \cong \text{Im}(F_0 \rightarrow F_{-1})$ .

### 3. GORENSTEIN DERIVING $\text{Hom}_R(-, -)$

We now return to categories of *modules*. We use  $\tilde{\mathcal{G}}\mathcal{P}$ ,  $\tilde{\mathcal{G}}\mathcal{I}$  and  $\tilde{\mathcal{G}}\mathcal{F}$  to denote the class of  $R$ -modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently,  $\mathcal{G}\mathcal{P}$ -precovers are always surjective, and  $\tilde{\mathcal{G}}\mathcal{P}$  contains all modules with finite projective dimension.

We now consider the functor  $\text{Hom}_R(-, -): \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A}$ , together with the categories

$$\mathcal{X} = \mathcal{G}\mathcal{P}, \tilde{\mathcal{X}} = \tilde{\mathcal{G}}\mathcal{P} \quad \text{and} \quad \mathcal{Y} = \mathcal{G}\mathcal{I}, \tilde{\mathcal{Y}} = \tilde{\mathcal{G}}\mathcal{I}.$$

In this case we define, in the sense of section 2.4,

$$\text{Ext}_{\mathcal{G}\mathcal{P}}^n(-, N) = \text{R}_{\mathcal{G}\mathcal{P}}^n \text{Hom}_R(-, N) \quad \text{and} \quad \text{Ext}_{\mathcal{G}\mathcal{I}}^n(M, -) = \text{R}_{\mathcal{G}\mathcal{I}}^n \text{Hom}_R(M, -),$$

for fixed  $R$ -modules  $M$  and  $N$ . We wish, of course, to apply Theorem 2.6 to this situation. Note that by [12, Proposition 2.18], we have:

**Proposition 3.1.** *If  $M$  is an  $R$ -module with  $\text{Gpd}_R M < \infty$ , then there exists a short exact sequence  $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$ , where  $G \rightarrow M$  is a  $\mathcal{G}\mathcal{P}$ -precover of  $M$  (please see Remark 2.3), and  $\text{pd}_R K = \text{Gpd}_R M - 1$  (in the case where  $M$  is Gorenstein projective, this should be interpreted as  $K = 0$ ).*

*Consequently, every  $R$ -module with finite Gorenstein projective dimension has a proper left  $\mathcal{G}\mathcal{P}$ -resolution (that is, there is an inclusion  $\tilde{\mathcal{G}}\mathcal{P} \subseteq \text{LeftRes}_{\mathcal{M}}(\mathcal{G}\mathcal{P})$ ).*

Furthermore, we will need the following from [12, Theorem 2.13]:

**Theorem 3.2.** *Let  $M$  be any  $R$ -module with  $\text{Gpd}_R M < \infty$ . Then*

$$\text{Gpd}_R M = \sup\{n \geq 0 \mid \text{Ext}_R^n(M, L) \neq 0 \text{ for some } R\text{-module } L \text{ with } \text{pd}_R L < \infty\}.$$

*Remark 3.3.* It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an  $R$ -module  $M$  is given by

$$\text{pd}_R M = \{n \geq 0 \mid \text{Ext}_R^n(M, L) \neq 0 \text{ for some } R\text{-module } L\}.$$

It also follows that if  $\text{pd}_R M < \infty$ , then every projective resolution of  $M$  is actually a proper left  $\mathcal{GP}$ -resolution of  $M$ .

**Lemma 3.4.** *Assume that  $M$  is an  $R$ -module with finite Gorenstein projective dimension, and let  $\mathbf{G}^+ = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$  be an augmented proper left  $\mathcal{GP}$ -resolution of  $M$  (which exists by Proposition 3.1). Then  $\text{Hom}_R(\mathbf{G}^+, H)$  is exact for all Gorenstein injective modules  $H$ .*

*Proof.* We split the proper resolution  $\mathbf{G}^+$  into short exact sequences. Hence it suffices to show exactness of  $\text{Hom}_R(\mathbf{S}, H)$  for all Gorenstein injective modules  $H$  and all short exact sequences

$$\mathbf{S} = 0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0,$$

where  $G \rightarrow M$  is a  $\mathcal{GP}$ -precover of some module  $M$  with  $\text{Gpd}_R M < \infty$  (recall that  $\mathcal{GP}$ -precovers are always surjective). By Proposition 3.1, there is a special short exact sequence,

$$\mathbf{S}' = 0 \longrightarrow K' \xrightarrow{\iota} G' \xrightarrow{\pi} M \longrightarrow 0,$$

where  $\pi: G' \rightarrow M$  is a  $\mathcal{GP}$ -precover and  $\text{pd}_R K' < \infty$ .

It is easy to see (as in Proposition 2.2) that the complexes  $\mathbf{S}$  and  $\mathbf{S}'$  are homotopy equivalent, and thus so are the complexes  $\text{Hom}_R(\mathbf{S}, H)$  and  $\text{Hom}_R(\mathbf{S}', H)$  for every (Gorenstein injective) module  $H$ . Hence it suffices to show the exactness of  $\text{Hom}_R(\mathbf{S}', H)$  whenever  $H$  is Gorenstein injective.

Now let  $H$  be any Gorenstein injective module. We need to prove the exactness of

$$\text{Hom}_R(G', H) \xrightarrow{\text{Hom}_R(\iota, H)} \text{Hom}_R(K', H) \longrightarrow 0.$$

To see this, let  $\alpha: K' \rightarrow H$  be any homomorphism. We wish to find  $\varrho: G' \rightarrow H$  such that  $\varrho\iota = \alpha$ . Now pick an exact sequence

$$0 \longrightarrow \tilde{H} \longrightarrow E \xrightarrow{g} H \longrightarrow 0,$$

where  $E$  is injective, and  $\tilde{H}$  is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines  $H$ ). Since  $\tilde{H}$  is Gorenstein injective and  $\text{pd}_R K' < \infty$ , we get  $\text{Ext}_R^1(K', \tilde{H}) = 0$  by [7, Lemma 1.3], and thus a lifting  $\varepsilon: K' \rightarrow E$  with  $g\varepsilon = \alpha$ :

$$\begin{array}{ccc} & K' & \xrightarrow{\iota} G' \\ & \swarrow \alpha & \downarrow \varepsilon \\ H & \xleftarrow{g} E & \end{array} \quad \begin{array}{c} \nearrow \tilde{\varepsilon} \\ \downarrow \varepsilon \\ \end{array}$$

Next, injectivity of  $E$  gives  $\tilde{\varepsilon}: G' \rightarrow E$  with  $\tilde{\varepsilon}\iota = \varepsilon$ . Now  $\varrho = g\tilde{\varepsilon}: G' \rightarrow H$  is the desired map. □

With a similar proof we get:

**Lemma 3.5.** *Assume that  $N$  is an  $R$ -module with finite Gorenstein injective dimension, and let  $\mathbf{H}^+ = 0 \rightarrow N \rightarrow H^0 \rightarrow H^1 \rightarrow \dots$  be an augmented proper right  $\mathcal{GI}$ -resolution of  $N$  (which exists by the dual of Proposition 3.1). Then  $\text{Hom}_R(G, \mathbf{H}^+)$  is exact for all Gorenstein projective modules  $G$ .  $\square$*

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

**Theorem 3.6.** *For all  $R$ -modules  $M$  and  $N$  with  $\text{Gpd}_R M < \infty$  and  $\text{Gid}_R N < \infty$ , we have isomorphisms*

$$\text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_{\mathcal{GI}}^n(M, N),$$

which are functorial in  $M$  and  $N$ .  $\square$

**3.7 (Definition of GExt).** Let  $M$  and  $N$  be  $R$ -modules with  $\text{Gpd}_R M < \infty$  and  $\text{Gid}_R N < \infty$ . Then we write

$$\text{GExt}_R^n(M, N) := \text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_{\mathcal{GI}}^n(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare GExt with the classical Ext. This is done in:

**Theorem 3.8.** *Let  $M$  and  $N$  be any  $R$ -modules. Then the following conclusions hold:*

(i) *There are natural isomorphisms  $\text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_R^n(M, N)$  under each of the conditions*

$$(\dagger) \text{ pd}_R M < \infty \quad \text{or} \quad (\ddagger) M \in \text{LeftRes}_{\mathcal{M}}(\mathcal{GP}) \text{ and } \text{id}_R N < \infty.$$

(ii) *There are natural isomorphisms  $\text{Ext}_{\mathcal{GI}}^n(M, N) \cong \text{Ext}_R^n(M, N)$  under each of the conditions*

$$(\dagger) \text{ id}_R N < \infty \quad \text{or} \quad (\ddagger) N \in \text{RightRes}_{\mathcal{M}}(\mathcal{GI}) \text{ and } \text{pd}_R M < \infty.$$

(iii) *Assume that  $\text{Gpd}_R M < \infty$  and  $\text{Gid}_R N < \infty$ . If either  $\text{pd}_R M < \infty$  or  $\text{id}_R N < \infty$ , then*

$$\text{GExt}_R^n(M, N) \cong \text{Ext}_R^n(M, N)$$

is functorial in  $M$  and  $N$ .

*Proof.* (i) Assume that  $\text{pd}_R M < \infty$ , and pick any projective resolution  $\mathbf{P}$  of  $M$ . By Remark 3.3,  $\mathbf{P}$  is also a proper left  $\mathcal{GP}$ -resolution of  $M$ , and thus

$$\text{Ext}_{\mathcal{GP}}^n(M, N) = \text{H}^n(\text{Hom}_R(\mathbf{P}, N)) = \text{Ext}_R^n(M, N).$$

In the case where  $M \in \text{LeftRes}_{\mathcal{M}}(\mathcal{GP})$  and  $\text{id}_R N = m < \infty$ , we see that Gorenstein projective modules are acyclic for the functor  $\text{Hom}_R(-, N)$ , that is,  $\text{Ext}_R^i(G, N) = 0$  (the usual Ext) for every Gorenstein projective module  $G$ , and every integer  $i > 0$ .

This is because, if  $G$  is a Gorenstein projective module, and  $i > 0$  is an integer, then there exists an exact sequence  $0 \rightarrow G \rightarrow Q^0 \rightarrow \dots \rightarrow Q^{m-1} \rightarrow C \rightarrow 0$ , where  $Q^0, \dots, Q^{m-1}$  are projective modules. Breaking this exact sequence into short exact ones, and applying  $\text{Hom}_R(-, N)$ , we get  $\text{Ext}_R^i(G, N) \cong \text{Ext}_R^{m+i}(C, N) = 0$ , as claimed.

Therefore [11, Chapter III, Proposition 1.2A] implies that  $\text{Ext}_R^n(-, N)$  can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of  $\text{GExt}_R^n(-, -)$ .  $\square$



4. GORENSTEIN DERIVING  $-\otimes_R-$

In dealing with the tensor product we need, of course, both left and right  $R$ -modules. Thus the following addition to Notation 1.1 is needed:

If  $\mathcal{C}$  is any of the categories in Notation 1.1 ( $\mathcal{M}, \mathcal{GP}$ , etc.), we write  ${}_R\mathcal{C}$ , respectively,  $\mathcal{C}_R$ , for the category of left, respectively, right,  $R$ -modules with the property describing the modules in  $\mathcal{C}$ .

Now we consider the functor  $-\otimes_R-: \mathcal{M}_R \times {}_R\mathcal{M} \rightarrow \mathcal{A}$ . For fixed  $M \in \mathcal{M}_R$  and  $N \in {}_R\mathcal{M}$  we define, in the sense of section 2.4:

$$\mathrm{Tor}_n^{\mathcal{GP}_R}(-, N) := L_n^{\mathcal{GP}_R}(-\otimes_R N) \quad \text{and} \quad \mathrm{Tor}_n^{{}_R\mathcal{GP}}(M, -) := L_n^{{}_R\mathcal{GP}}(M \otimes_R -),$$

together with

$$\mathrm{Tor}_n^{\mathcal{GF}_R}(-, N) := L_n^{\mathcal{GF}_R}(-\otimes_R N) \quad \text{and} \quad \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M, -) := L_n^{{}_R\mathcal{GF}}(M \otimes_R -).$$

The first two Tors use proper left Gorenstein *projective* resolutions, and the last two Tors use proper left Gorenstein *flat* resolutions. In order to compare these different Tors, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of

$$(\mathcal{X}, \tilde{\mathcal{X}}) = (\mathcal{GP}_R, \tilde{\mathcal{GP}}_R) \quad \text{or} \quad (\mathcal{GF}_R, \tilde{\mathcal{GF}}_R),$$

and

$$(\mathcal{Y}, \tilde{\mathcal{Y}}) = ({}_R\mathcal{GP}, {}_R\tilde{\mathcal{GP}}) \quad \text{or} \quad ({}_R\mathcal{GF}, {}_R\tilde{\mathcal{GF}}),$$

namely, the covariant-covariant version of Theorem 2.6, instead of the stated contra-variant-covariant version. We will need the classical notion:

**Definition 4.1.** The *left finitistic projective dimension*  $\mathrm{LeftFPD}(R)$  of  $R$  is defined as

$$\mathrm{LeftFPD}(R) = \sup\{\mathrm{pd}_R M \mid M \text{ is a left } R\text{-module with } \mathrm{pd}_R M < \infty\}.$$

The right finitistic projective dimension  $\mathrm{RightFPD}(R)$  of  $R$  is defined similarly.

*Remark 4.2.* When  $R$  is commutative and Noetherian, the dimensions  $\mathrm{LeftFPD}(R)$  and  $\mathrm{RightFPD}(R)$  coincide and are equal to the Krull dimension of  $R$ , by [10, Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12, Proposition 3.3], [12, Theorem 3.5] and [12, Proposition 3.18], respectively:

**Proposition 4.3.** *If  $R$  is right coherent with finite  $\mathrm{LeftFPD}(R)$ , then every Gorenstein projective left  $R$ -module is also Gorenstein flat. That is, there is an inclusion  ${}_R\mathcal{GP} \subseteq {}_R\mathcal{GF}$ .  $\square$*

**Theorem 4.4.** *For any left  $R$ -module  $M$ , we consider the following three conditions:*

- (i) *The left  $R$ -module  $M$  is  $G$ -flat.*
- (ii) *The Pontryagin dual  $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  (which is a right  $R$ -module) is  $G$ -injective.*
- (iii)  *$M$  has an augmented proper right resolution  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$  consisting of flat left  $R$ -modules, and  $\mathrm{Tor}_i^R(I, M) = 0$  for all injective right  $R$ -modules  $I$ , and all  $i > 0$ .*

*The implication (i)  $\Rightarrow$  (ii) always holds. If  $R$  is right coherent, then also (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i), and hence all three conditions are equivalent.  $\square$*

**Proposition 4.5.** *Assume that  $R$  is right coherent. If  $M$  is a left  $R$ -module with  $\text{Gfd}_R M < \infty$ , then there exists a short exact sequence  $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$ , where  $G \rightarrow M$  is an  ${}_R\mathcal{GF}$ -precover of  $M$ , and  $\text{fd}_R K = \text{Gfd}_R M - 1$  (in the case where  $M$  is Gorenstein flat, this should be interpreted as  $K = 0$ ).*

*In particular, every left  $R$ -module with finite Gorenstein flat dimension has a proper left  ${}_R\mathcal{GF}$ -resolution (that is, there is an inclusion  ${}_R\widetilde{\mathcal{GF}} \subseteq \text{LeftRes}_R \mathcal{M}({}_R\mathcal{GF})$ ).* □

Our first result is:

**Lemma 4.6.** *Let  $M$  be a left  $R$ -module with  $\text{Gpd}_R M < \infty$ , and let  $\mathbf{G}^+ = \dots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$  be an augmented proper left  ${}_R\mathcal{GP}$ -resolution of  $M$  (which exists by Proposition 3.1). Then the following conclusions hold:*

- (i)  $T \otimes_R \mathbf{G}^+$  is exact for all Gorenstein flat right  $R$ -modules  $T$ .
- (ii) If  $R$  is left coherent with finite  $\text{RightFPD}(R)$ , then  $T \otimes_R \mathbf{G}^+$  is exact for all Gorenstein projective right  $R$ -modules  $T$ .

*Proof.* (i) By Theorem 4.4 above, the Pontryagin dual  $H = \text{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$  is a Gorenstein injective left  $R$ -module. Hence  $\text{Hom}_R(\mathbf{G}^+, H) \cong \text{Hom}_{\mathbb{Z}}(T \otimes_R \mathbf{G}^+, \mathbb{Q}/\mathbb{Z})$  is exact by Proposition 3.4. Since  $\mathbb{Q}/\mathbb{Z}$  is a faithfully injective  $\mathbb{Z}$ -module,  $T \otimes_R \mathbf{G}^+$  is exact too.

(ii) With the given assumptions on  $R$ , the dual of Proposition 4.3 implies that every Gorenstein projective right  $R$ -module also is Gorenstein flat. □

**Lemma 4.7.** *Assume that  $R$  is right coherent with finite  $\text{LeftFPD}(R)$ . Let  $M$  be a left  $R$ -module with  $\text{Gfd}_R M < \infty$ , and let  $\mathbf{G}^+ = \dots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$  be an augmented proper left  ${}_R\mathcal{GF}$ -resolution of  $M$  (which exists by Proposition 4.5, since  $R$  is right coherent). Then the following conclusions hold:*

- (i)  $\text{Hom}_R(\mathbf{G}^+, H)$  is exact for all Gorenstein injective left  $R$ -modules  $H$ .
- (ii)  $T \otimes_R \mathbf{G}^+$  is exact for all Gorenstein flat right  $R$ -modules  $T$ .
- (iii) If  $R$  is also left coherent with finite  $\text{RightFPD}(R)$ , then  $T \otimes_R \mathbf{G}^+$  is exact for all Gorenstein projective right  $R$ -modules  $T$ .

*Proof.* (i) Since  $\text{Gfd}_R M < \infty$  and  $R$  is right coherent, Proposition 4.5 gives a special short exact sequence  $0 \rightarrow K' \rightarrow G' \rightarrow M \rightarrow 0$ , where  $G' \rightarrow M$  is an  ${}_R\mathcal{GF}$ -precover of  $M$ , and  $\text{fd}_R K' < \infty$ . Since  $R$  has  $\text{LeftFPD}(R) < \infty$ , [13, Proposition 6] implies that also  $\text{pd}_R K' < \infty$ . Now the proof of Lemma 3.4 applies.

(ii) If  $T$  is a Gorenstein flat right  $R$ -module, then the left  $R$ -module  $H = \text{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$  is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of

$$\text{Hom}_R(\mathbf{G}^+, H) \cong \text{Hom}_{\mathbb{Z}}(T \otimes_R \mathbf{G}^+, \mathbb{Q}/\mathbb{Z}).$$

Since  $\mathbb{Q}/\mathbb{Z}$  is a faithfully injective  $\mathbb{Z}$ -module, we also have exactness of  $T \otimes_R \mathbf{G}^+$ , as desired.

(iii) Under the extra assumptions on  $R$ , the dual of Proposition 4.3 implies that every Gorenstein projective right  $R$ -module is also Gorenstein flat. Thus (iii) follows from (ii). □

**Theorem 4.8.** *Assume that  $R$  is both left and right coherent, and that both  $\text{LeftFPD}(R)$  and  $\text{RightFPD}(R)$  are finite. For every right  $R$ -module  $M$ , and every left  $R$ -module  $N$ , the following conclusions hold:*

(i) If  $\text{Gfd}_R M < \infty$  and  $\text{Gfd}_R N < \infty$ , then

$$\text{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GF}}(M, N).$$

(ii) If  $\text{Gpd}_R M < \infty$  and  $\text{Gfd}_R N < \infty$ , then

$$\text{Tor}_n^{\mathcal{GP}_R}(M, N) \cong \text{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GF}}(M, N).$$

(iii) If  $\text{Gfd}_R M < \infty$  and  $\text{Gpd}_R N < \infty$ , then

$$\text{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GP}}(M, N) \cong \text{Tor}_n^{R\mathcal{GF}}(M, N).$$

(iv) If  $\text{Gpd}_R M < \infty$  and  $\text{Gpd}_R N < \infty$ , then

$$\text{Tor}_n^{\mathcal{GP}_R}(M, N) \cong \text{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GP}}(M, N) \cong \text{Tor}_n^{R\mathcal{GF}}(M, N).$$

All the isomorphisms are functorial in  $M$  and  $N$ .

*Proof.* Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6.  $\square$

**4.9 (Definition of  $g\text{Tor}$  and  $\text{GTor}$ ).** Assume that  $R$  is both left and right coherent, and that both  $\text{LeftFPD}(R)$  and  $\text{RightFPD}(R)$  are finite. Furthermore, let  $M$  be a right  $R$ -module, and let  $N$  be a left  $R$ -module. If  $\text{Gfd}_R M < \infty$  and  $\text{Gfd}_R N < \infty$ , then we write

$$g\text{Tor}_n^R(M, N) := \text{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GF}}(M, N)$$

for the isomorphic abelian groups in Theorem 4.8(i). If  $\text{Gpd}_R M < \infty$  and  $\text{Gpd}_R N < \infty$ , then we write

$$\text{GTor}_n^R(M, N) := \text{Tor}_n^{\mathcal{GP}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GP}}(M, N)$$

for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8:

**Theorem 4.10.** *Assume that  $R$  is both left and right coherent, and that both  $\text{LeftFPD}(R)$  and  $\text{RightFPD}(R)$  are finite. For every right  $R$ -module  $M$  with finite  $\text{Gpd}_R M$ , and for every left  $R$ -module  $N$  with  $\text{Gpd}_R N < \infty$ , we have isomorphisms:*

$$g\text{Tor}_n^R(M, N) \cong \text{GTor}_n^R(M, N)$$

that are functorial in  $M$  and  $N$ .

Finally we compare  $g\text{Tor}$  (and hence  $\text{GTor}$ ) with the usual  $\text{Tor}$ .

**Theorem 4.11.** *Assume that  $R$  is both left and right coherent, and that both  $\text{LeftFPD}(R)$  and  $\text{RightFPD}(R)$  are finite. Furthermore, let  $M$  be a right  $R$ -module with  $\text{Gfd}_R M < \infty$ , and let  $N$  be a left  $R$ -module with  $\text{Gfd}_R N < \infty$ . If either  $\text{fd}_R M < \infty$  or  $\text{fd}_R N < \infty$ , then there are isomorphisms*

$$g\text{Tor}_n^R(M, N) \cong \text{Tor}_n^R(M, N)$$

that are functorial in  $M$  and  $N$ .

*Proof.* If  $\text{fd}_R M < \infty$ , then we also have  $\text{pd}_R M < \infty$  by [13, Proposition 6] (since  $\text{RightFPD}(R) < \infty$ ). Let  $\mathbf{P}$  be any projective resolution of  $M$ . As noted in Remark 3.3,  $\mathbf{P}$  is also a proper left  $\mathcal{GP}_R$ -resolution of  $M$ . Hence, Theorem 4.8(ii) and the definitions give:

$$g\text{Tor}_n^R(M, N) = \text{Tor}_n^{\mathcal{GP}_R}(M, N) = \text{H}_n(\mathbf{P} \otimes_R N) = \text{Tor}_n^R(M, N),$$

as desired.  $\square$

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