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# WHICH FINITE SIMPLE GROUPS ARE UNIT GROUPS?

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**ABSTRACT.** We prove that if  $G$  is a finite simple group which is the unit group of a ring, then  $G$  is isomorphic to either (a) a cyclic group of order 2; (b) a cyclic group of prime order  $2^k - 1$  for some  $k$ ; or (c) a projective special linear group  $\mathrm{PSL}_n(\mathbb{F}_2)$  for some  $n \geq 3$ . Moreover, these groups do all occur as unit groups. We deduce this classification from a more general result, which holds for groups  $G$  with no non-trivial normal 2-subgroup.

Throughout this paper, rings will be assumed to be unital, but not necessarily commutative, and ring homomorphisms send 1 to 1. The finite groups  $G$  of odd order which occur as unit groups of rings were determined in [3]. We will prove similar results for a more general class of groups; the description of this class of groups uses the following.

**Definition 1.** For a finite group  $G$ , the  $p$ -core of  $G$  is the largest normal  $p$ -subgroup of  $G$ . We denote this subgroup by  $O_p(G)$ . It is the intersection of all Sylow  $p$ -subgroups of  $G$ .

We now state the main result. The authors<sup>1</sup> are most grateful to the anonymous referee for our earlier paper [2], who recognized that one of the results proved in that paper could be strengthened into the following.

**Theorem 2.** Let  $G$  denote a finite group such that  $O_2(G) = \{1\}$  and such that  $G$  is isomorphic to the unit group of a ring  $R$ . Then

$$G \cong \mathrm{GL}_{n_1}(\mathbb{F}_{2^{k_1}}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{F}_{2^{k_r}}).$$

Before proving Theorem 2, we record the following corollary.

**Corollary 3.** The finite simple groups which occur as unit groups of rings are precisely the groups

- (a)  $\mathbb{Z}/2\mathbb{Z}$ ,
- (b)  $\mathbb{Z}/p\mathbb{Z}$  for a Mersenne prime  $p = 2^k - 1$ ,
- (c)  $\mathrm{PSL}_n(\mathbb{F}_2)$  for  $n \geq 3$ .

*Proof.* If  $G$  is a finite simple group, then either  $O_2(G) = \{1\}$  or  $O_2(G) = G$ . If  $O_2(G) = G$ , then  $G$  is a 2-group, and because we are assuming  $G$  is simple, we must have  $G \cong \mathbb{Z}/2\mathbb{Z}$ , which for instance is isomorphic to the unit group of  $\mathbb{Z}$ .

Hence assume  $G$  is a finite simple group which is isomorphic to the unit group of a ring and further assume  $O_2(G) = \{1\}$ . By Theorem 2, we know

$$G \cong \mathrm{GL}_{n_1}(\mathbb{F}_{2^{k_1}}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{F}_{2^{k_r}}).$$

These groups all occur as unit groups of the corresponding products of matrix rings, so we are reduced to determining which of them are simple; this forces

$$G \cong \mathrm{GL}_n(\mathbb{F}_{2^k}).$$

If  $n > 1$  and  $k > 1$ , then the subgroup of invertible scalar matrices forms a nontrivial normal subgroup. Hence two possibilities remain. If  $n = 1$ , then  $\mathrm{GL}_1(\mathbb{F}_{2^k})$  is cyclic of order  $2^k - 1$ ; such a group is simple if and only if its order is prime. If  $k = 1$ , then  $\mathrm{GL}_n(\mathbb{F}_2) = \mathrm{PSL}_n(\mathbb{F}_2)$ . For the case  $k = 1, n = 2$ , we have  $\mathrm{PSL}_2(\mathbb{F}_2) \cong S_3$  (see for example [4, Section 3.3.1]); this group is not simple. For the cases  $k = 1, n \geq 3$ , it is well-known that  $\mathrm{PSL}_n(\mathbb{F}_2)$  is simple (see for example [4, Section 3.3.2]). This completes the proof.  $\square$

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**Remark 4.** The simple groups  $A_8$  and  $\mathrm{PSL}_2(\mathbb{F}_7)$  also occur as unit groups. This follows immediately from the exceptional isomorphisms

$$A_8 \cong \mathrm{PSL}_4(\mathbb{F}_2) \text{ and } \mathrm{PSL}_2(\mathbb{F}_7) \cong \mathrm{PSL}_3(\mathbb{F}_2).$$

See for instance [4, Section 3.12].

Having recorded the above consequences of the main result, we now gather the preliminary results used in its proof. We begin with the following observation.

**Lemma 5.** Let  $G$  denote a finite group with  $O_2(G) = \{1\}$ , and let  $R$  denote a ring with  $R^\times \cong G$ . Then  $R$  has characteristic 2.

*Proof.* The elements 1 and  $-1$  are units in  $R$  and are in the center of  $R$ , hence are in the center of  $R^\times$ . By the assumption  $O_2(G) = \{1\}$ , the center of  $G$  cannot contain any elements of order 2. Hence  $1 = -1$ .  $\square$

**Lemma 6.** Keep notation as in Lemma 5, and fix an isomorphism  $R^\times \cong G$ . Because  $R$  has characteristic 2, we have a natural map

$$\varphi : \mathbb{F}_2[G] \rightarrow R$$

extending the fixed embedding of  $G$  into  $R$ . The image of  $\varphi$  is a ring with unit group isomorphic to  $G$ .

*Proof.* Write  $S$  for the image of  $\varphi$ . On one hand, we have that  $S^\times \subseteq R^\times \cong G$ . On the other hand, the induced map  $\varphi : G \rightarrow S^\times \rightarrow R^\times$  is surjective. This shows that the unit group of  $S$  is isomorphic to  $G$ .  $\square$

**Lemma 7.** Let  $R$  denote a finite ring of characteristic 2. If  $J \subseteq R$  is a two-sided ideal such that  $J^2 = 0$ , then  $1 + J$  is a normal elementary abelian 2-subgroup of  $R^\times$ .

*Proof.* Note that for any  $j, k \in J$  and  $r \in R^\times$ , we have

- $(1 + j)^2 = 1 + j^2 = 1$ ;
- $(1 + j)(1 + k) = 1 + j + k + jk = 1 + j + k = (1 + k)(1 + j)$ ;
- $r(1 + j)r^{-1} = 1 + rjr^{-1} \in 1 + J$ .

The first of these calculations shows that  $1 + J$  is a subset of  $R^\times$ , and the three calculations together show that it is a normal elementary abelian 2-group.  $\square$

We now use these preliminary results to prove our main theorem.

*Proof of Theorem 2.* By Lemma 6, we may assume  $R$  is a finite ring (and is in particular artinian) and has characteristic 2. Let  $J$  denote a two-sided ideal of  $R$  such that  $J^2 = 0$ . By Lemma 7, the set  $1 + J$  is a normal 2-subgroup of  $R^\times$ , and so by the assumption  $O_2(G) = \{1\}$ , we have  $J = \{0\}$ . Thus the ring  $R$  has no non-zero two-sided ideals  $J$  with  $J^2 = 0$ , and hence  $R$  has no non-zero two-sided nilpotent ideals. By [1, Theorem 5.4.5], the artinian ring  $R$  is semisimple. By Wedderburn's Theorem [1, Theorem 5.3.4], we have

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

for some  $n_1, \dots, n_r \geq 1$  and some division algebras  $D_1, \dots, D_r$ . Our ring  $R$  is finite and hence each  $D_i$  is finite. By another theorem of Wedderburn [1, Theorem 3.8.6], we have that each  $D_i$  is a finite field. Finally, because the ring  $R$  has characteristic 2, each field  $D_i$  has characteristic 2. This completes the proof.  $\square$

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